

A Perspective on Constructive Quantum Field Theory

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Abstract

An overview of the accomplishments of constructive quantum field theory is provided.¹

1 Introduction: Background and Motivations

Quantum field theory (QFT) is widely viewed as one of the most successful theories in science — it has predicted phenomena before they were observed in nature², and its predictions are believed to be confirmed by experiments to within an extraordinary degree of accuracy³. Though it has undergone a long and complex development from its origins⁴ in the 1929/30 papers of Heisenberg and Pauli [105, 106] and has attained an ever increasing theoretical sophistication, it is still not clear in which sense the physically central quantum field theories such as quantum electrodynamics (QED), quantum chromodynamics (QCD) and the Standard Model (SM) are mathematically well defined theories based upon fundamental physical principles that go beyond the merely *ad hoc*. Needless to say, there are many physicists working with quantum field theories for whom the question is of little

¹This is an expanded version of an article commissioned for UNESCO's Encyclopedia of Life Support Systems (EOLSS).

²For example, the existence and properties of the W and Z bosons, as well as the top and charm quarks, were predicted before they were found experimentally.

³For example, the two parts in one billion difference between the theoretical prediction from the Standard Model and the experimentally measured value of the anomalous magnetic moment of the muon [44].

⁴A certain amount of arbitrariness and personal taste must go into pointing to a single point of origin, since the 1927 discussion of a quantum theory of electromagnetic radiation by Dirac as well as the studies of relativistic wave mechanics by Dirac, Schrödinger and even de Broglie were influential. In any case, the interested reader should see [169] for a detailed account of the birth of QFT.

to no interest. But there are also many who are not satisfied with the conceptual/mathematical state of quantum field theory and have dedicated entire careers to an attempt to attain some clarity in the matter.

This is not the place to explain the grounds for this dissatisfaction; instead, the goal of this paper is to provide a perspective on “constructive quantum field theory” (CQFT), the subfield of mathematical physics concerned with establishing the existence of concrete models of relativistic quantum field theory in a very precise mathematical sense and then studying their properties from the point of view of both mathematics and physics. Although the insights and techniques won by the constructive quantum field theorists have proven to be useful also in statistical mechanics and many body physics, these further successes of CQFT are not discussed here. In addition, we shall restrict our attention solely to relativistic QFT on d dimensional Minkowski space, $d \geq 2$; to this point in time most work in CQFT has been carried out precisely in that context. Throughout, as is customary in QFT, we adopt physical units in which $c = \hbar/2\pi = 1$.

In the 1950’s and early 1960’s various “axiomatizations” of QFT were formulated. These can be seen to have two primary goals — (1) to abstract from heuristic QFT the fundamental principles of QFT and to formulate them in a mathematically precise framework; (2) on the basis of this framework, to formulate and solve conceptual and mathematical problems of heuristic QFT in a mathematically rigorous manner. As it turned out, the study and further development of these axiom systems led to unanticipated conceptual and physical breakthroughs and insights, but these are also not our topic here.

The first and most narrow axiomatization scheme of the two briefly discussed here is constituted by the *Wightman axioms* (see *e.g.* [179]). This axiom system adheres most closely to heuristic QFT in that the basic objects are local, covariant fields acting on a fixed Hilbert space. A (scalar Bose) Wightman theory is a quadruple $(\phi, \mathcal{H}, U, \Omega)$ consisting of a Hilbert space \mathcal{H} , a strongly continuous unitary representation U of the (covering group of the) identity component \mathcal{P}_+^\uparrow of the Poincaré group acting upon \mathcal{H} , a unit vector $\Omega \in \mathcal{H}$ which spans the subspace of all vectors in \mathcal{H} left invariant by $U(\mathcal{P}_+^\uparrow)$,⁵ and an (unbounded) operator valued distribution⁶ ϕ such that for every test function f , the operator $\phi(f)$ has a dense invariant domain \mathcal{D} spanned by all products of field operators applied to Ω . These conditions are a rigorous formulation of tacit assumptions made in nearly all heuristic field theories. In addition, a number of fundamental principles were identified and formulated in this framework.

Relativistic Covariance: For every Poincaré element $(\Lambda, a) \in \mathcal{P}_+^\uparrow$ one has $U(\Lambda, a)\phi(x)U(\Lambda, a)^{-1} = \phi(\Lambda x + a)$, in the sense of operator valued distributions on \mathcal{D} .

⁵This condition, referred to as the “uniqueness of the vacuum,” is posited for convenience. With known techniques one can decompose a given model into submodels that satisfy this condition as well as the remaining conditions [7, 22, 50, 51].

⁶Although it is possible, indeed sometimes necessary, to choose other test function spaces, here we shall restrict our attention to the Schwartz tempered test function space $\mathcal{S}(\mathbb{R}^d)$.

Einstein Causality:⁷ For all spacelike separated $x, y \in \mathbb{R}^4$ one has $\phi(x)\phi(y) = \phi(y)\phi(x)$ in the sense of operator valued distributions on \mathcal{D} .

The Spectrum Condition (stability of the field system): Restricting one's attention to the translation subgroup $\mathbb{R}^4 \subset \mathcal{P}_+^\uparrow$, the spectrum of the self-adjoint generators of the group $U(\mathbb{R}^4)$ is contained in the closed forward lightcone $\overline{V}_+ = \{p = (p_0, p_1, p_2, p_3) \in \mathbb{R}^4 \mid p_0^2 - p_1^2 - p_2^2 - p_3^2 \geq 0\}$.

The reader is referred to [120, 179] for a discussion of the physical interpretation and motivation of these conditions. There is an equivalent formulation of these conditions in terms of the *Wightman functions* [179]

$$W_n(x_1, x_2, \dots, x_n) \equiv \langle \Omega, \phi(x_1)\phi(x_2) \cdots \phi(x_n)\Omega \rangle, \quad n \in \mathbb{N},$$

which are distributions on $\mathcal{S}(\mathbb{R}^{dn})$. These two sets of conditions are referred to collectively as the *Wightman axioms*. There are closely related sets of conditions for Fermi fields and higher spin Bose fields [120, 179].

A more general axiom system which is conceptually closer to the actual operational circumstances of a theory tested by laboratory experiments is constituted by the *Haag–Araki–Kastler axioms* (HAK axioms), also referred to as local quantum physics or algebraic quantum field theory (AQFT). Although more general formulations of AQFT are available, for the purposes of this paper it will suffice to limit our attention to a quadruple⁸ $(\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O} \in \mathcal{R}}, \mathcal{H}, U, \Omega)$ with \mathcal{H} , U and Ω as above and $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O} \in \mathcal{R}}$ a net of von Neumann algebras $\mathcal{A}(\mathcal{O})$ acting on \mathcal{H} , where \mathcal{O} ranges through a suitable set \mathcal{R} of nonempty open subsets of Minkowski space. The algebra $\mathcal{A}(\mathcal{O})$ is interpreted as the algebra generated by all (bounded) observables measurable in the spacetime region \mathcal{O} , so the net $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O} \in \mathcal{R}}$ is naturally assumed to satisfy isotony: if $\mathcal{O}_1 \subset \mathcal{O}_2$, then one must have $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$. In this framework the basic principles are formulated as follows.

Relativistic Covariance: For every Poincaré element $(\Lambda, a) \in \mathcal{P}_+^\uparrow$ and spacetime region $\mathcal{O} \in \mathcal{R}$ one has $U(\Lambda, a)\mathcal{A}(\mathcal{O})U(\Lambda, a)^{-1} = \mathcal{A}(\Lambda\mathcal{O} + a)$.

Einstein Causality:⁹ For all spacelike separated regions $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{R}$ one has $AB = BA$ for all $A \in \mathcal{A}(\mathcal{O}_1)$ and all $B \in \mathcal{A}(\mathcal{O}_2)$.

The Spectrum Condition (stability of the field system): Same as above.

The reader is referred to [9, 104] for a discussion of the physical interpretation and motivation of these conditions. The relation between the Wightman axioms and AQFT is well understood (see *e.g.* [23, 34, 52, 69, 183]). It is important to note that, in general, infinitely many different fields in the sense of the Wightman axioms are associated with the same net of observable algebras. Indeed, an analogy has

⁷Also called microscopic causality, local commutativity or, somewhat misleadingly, locality.

⁸In point of fact, these conditions actually describe an algebraic QFT in a (Minkowski space) *vacuum representation*. By no means is AQFT limited to such circumstances; some other representations of physical interest are briefly discussed below. Moreover, the algebraic approach to QFT has proven to be particularly fruitful in addressing conceptual and mathematical problems concerning quantum fields on curved spacetimes.

⁹Also often referred to as locality.

often been drawn between the choice of a particular coordinate system, made in order to carry out a computation more conveniently, in differential geometry and the choice of a particular field out of the many fields associated with a given net. For this and other reasons, those who work in mathematical QFT consider nets of observable algebras to be more *intrinsic* than the associated quantum fields, which are used primarily for computational convenience.

Associated to any Wightman system $(\phi, \mathcal{H}, U, \Omega)$ is a net of $*$ -algebras $\mathcal{P}(\mathcal{O})$, $\mathcal{O} \subset \mathbb{R}^4$. Because all field operators have the common, dense domain $\mathcal{D} \subset \mathcal{H}$, arbitrary “polynomials” of field operators can be formed on \mathcal{D} . $\mathcal{P}(\mathcal{O})$ denotes the algebra formed by all polynomials (in the sense of functions of infinitely many variables) in which the supports of all test functions of all field operators entering into the polynomial are contained in the spacetime region \mathcal{O} . The algebras $\mathcal{P}(\mathcal{O})$ are not C^* -algebras but satisfy all of the other HAK axioms. Despite the non-intrinsic nature of such algebras and despite the technical disadvantages of working with $*$ -algebras instead of with C^* -algebras, mathematical quantum field theorists find it convenient for various purposes to work with such nets or with similar nets of non- C^* -algebras.

The goal of constructive QFT, as is it usually understood, is to construct in a mathematically rigorous manner physically relevant quantum field models which satisfy one of these systems of axioms and then to study their mathematical properties with an emphasis on those properties which can be shown to have physical relevance. This article briefly describes such models and the means by which they were constructed and is organized both historically and by the construction techniques employed.

As pointed out independently by Borchers and Uhlmann, the Wightman axioms can be understood in a representation independent manner in terms of what is now called the Borchers (or Borchers–Uhlmann) algebra — a tensor algebra constructed out of the test function space $\mathcal{S}(\mathbb{R}^4)$ with operations directly motivated by the Wightman axioms. Borchers algebras have been extensively studied from the point of view of QFT, especially by Borchers, Uhlmann, Yngvason and Lassner (see *e.g.* [109] for definitions and references). A Wightman system can be thought of as a concrete representation of the Borchers algebra, and for a time there was hope one could arrive at quantum field models by defining suitable states on the Borchers algebra and employing the standard GNS construction to obtain the corresponding representation. However, it proved to be too difficult to conjure such states.

The first quantum field models constructed were the free quantum fields, the Wick powers of such free fields and the so-called generalized free fields. These models have been constructed using a variety of techniques (cf. *e.g.* [8, 12, 29, 95, 120, 145, 174, 196]) and have been shown to satisfy the two axiom systems discussed above; a recent construction of free fields which is of particular conceptual interest is briefly described in Section 6. The Hilbert space upon which such fields act is called the Fock space. Common to these models is the fact that their S -matrix,

the object which describes the scattering behavior of the “particles” described by such fields (*cf.* [9, 120]), is just the identity map.

We turn now to models with nontrivial S-matrices, *i.e.* interacting quantum field models. When referring to the models, we employ the standard notation M_d , which means quantum model M in d spacetime dimensions. Because the mathematical and conceptual difficulties inherent in the construction of quantum field models are quite daunting, constructive quantum field theorists proceeded by considering increasingly challenging models; this often entailed starting the study of the model M with $d = 2$, then $d = 3$, and finally $d = 4$. At this point in time only a few models have been constructed in four spacetime dimensions. In this respect, the reader is referred to Section 8 for a few words about the outlook for CQFT after nearly fifty years of strenuous effort. The reader should note that all results discussed in this article, unless explicitly stated otherwise, are proven according to the criteria accepted by mathematicians and not merely on the basis of the plausibility arguments accepted by most physicists as “proof”.

2 Algebraic Constructions I

Preceded by the 1965 dissertations of Jaffe [116] and Lanford [132], the first constructions of interacting quantum fields were carried out in the late 1960’s and early 1970’s. In this early work the real time models were constructed directly using operator algebras and functional analysis as the primary tools. Due to Haag’s Theorem, it was known that the Hilbert space in which these interacting quantum fields would be defined could not be Fock space (see *e.g.* [184]). However, because Fock space was the sole available starting point at that time, “cutoffs” were placed on the interacting theories so that they could be realized on Fock space in a mathematically meaningful manner. These cutoffs were of two general kinds — finite volume cutoffs and ultraviolet cutoffs — each addressing independent sources of the divergences known in QFT since early in its development. Guided by heuristic QFT’s division of Lagrangian quantum field models into superrenormalizable, renormalizable and nonrenormalizable models,¹⁰ the constructive quantum field theorists began with the simplest category, the superrenormalizable models. To be able to address the infinite volume divergence without wrestling simultaneously with the ultraviolet divergence, constructive quantum field theorists first considered self-interacting bosonic quantum field models in two spacetime dimensions.

We begin with Glimm and Jaffe’s construction of the $(\phi^4)_2$ model [81–84], the self-interacting scalar Bose field on two dimensional Minkowski space with Lagrangian self-interaction $\lambda\phi^4$, where λ is the coupling constant. Let \mathcal{H}_0 be the Fock space for a (free) scalar hermitian Bose field $\phi(t, x)$ of mass $m > 0$ ($(t, x) \in \mathbb{R}^2$). Let $\pi(t, x) = \partial\phi(t, x)/\partial t$ be the canonically conjugate momentum field and $\mathcal{D} \subset \mathcal{H}_0$ be the dense set of finite-particle vectors in \mathcal{H}_0 . Then, for every f in a dense subspace

¹⁰This classification is based upon the perturbation theory associated by Feynman and others with interacting fields, viewed as perturbations of free fields.

$\mathcal{S}(\mathbb{R})$ of $L^2(\mathbb{R})$, the operator $\phi_0(f) \equiv \int \phi(0, x) f(x) dx$ is essentially self-adjoint on \mathcal{D} and $\phi_0(f)\mathcal{D} \subset \mathcal{D}$ (similarly for $\pi_0(f)$). These operators satisfy the canonical commutation relations (CCR) on \mathcal{D} :

$$\begin{aligned}\phi_0(f)\pi_0(g) - \pi_0(g)\phi_0(f) &= i \langle f, g \rangle \mathbb{I}, \\ \phi_0(f)\phi_0(g) - \phi_0(g)\phi_0(f) &= 0 = \pi_0(f)\pi_0(g) - \pi_0(g)\pi_0(f),\end{aligned}$$

for all $f, g \in \mathcal{S}(\mathbb{R})$, where $\langle \cdot, \cdot \rangle$ is the inner product on $L^2(\mathbb{R})$ and \mathbb{I} is the identity operator on \mathcal{H} . When exponentiated using the spectral calculus, (the closures of) these operators provide a Weyl representation of the CCR. For each bounded open subset $\mathbf{O} \subset \mathbb{R}$, denote by $\mathcal{A}(\mathbf{O})$ the von Neumann algebra generated by the Weyl unitaries

$$\{e^{i\phi_0(f)}, e^{i\pi_0(f)} \mid f \in \mathcal{S}(\mathbb{R}), \text{supp}(f) \subset \mathbf{O}\}.$$

($\text{supp}(f)$ denotes the support of the function f .)

The total energy

$$H_0 = \frac{1}{2} \int : (\pi(0, x)^2 + \nabla \phi(0, x)^2 + m^2 \phi(0, x)^2) : dx$$

of the free field is a positive quadratic form on $\mathcal{D} \times \mathcal{D}$ and therefore determines uniquely a positive self-adjoint operator, which we also denote by H_0 . The double colons indicate that the expression between them is Wick ordered, which is a physically motivated way to define in a rigorous manner a product of operator valued distributions. In this case, the Wick ordering is performed with respect to the Fock vacuum (*cf.* [90]). With $g \in L^2(\mathbb{R})$ nonnegative of compact support, Glimm and Jaffe showed that, for each $\lambda > 0$, the cut-off interacting Hamilton operator

$$H(g) \equiv H_0 + \lambda \int : \phi(0, x)^4 : g(x) dx$$

is essentially self-adjoint on \mathcal{D} ,¹¹ and its self-adjoint closure, also denoted by $H(g)$, is bounded from below. By adding a suitable multiple of the identity we may take 0 to be the minimum of its spectrum. Then, they proved that 0 is a simple eigenvalue of $H(g)$ with normalized eigenvector $\Omega(g) \in \mathcal{H}_0$.

For any $t \in \mathbb{R}$, let \mathbf{O}_t denote the subset of \mathbb{R} consisting of all points with distance less than $|t|$ to \mathbf{O} . By choosing the cutoff function g to be equal to 1 on \mathbf{O}_t , then for any $A \in \mathcal{A}(\mathbf{O})$ the operator

$$\sigma_t(A) \equiv e^{itH(g)} A e^{-itH(g)}$$

is independent of g and is contained in $\mathcal{A}(\mathbf{O}_t)$. For any bounded open $\mathcal{O} \subset \mathbb{R}^2$ and $t \in \mathbb{R}$, let $\mathbf{O}(t) = \{x \in \mathbb{R} \mid (t, x) \in \mathcal{O}\}$ be the time t slice of \mathcal{O} . We define $\mathcal{A}(\mathcal{O})$

¹¹Without the cutoff g , the interacting Hamilton operator is *not* densely defined in Fock space.

to be the von Neumann algebra generated by $\bigcup_s \sigma_s(\mathcal{A}(\mathcal{O}(s)))$.¹² Finally, we let \mathcal{A} denote the closure in the operator norm of the union $\bigcup \mathcal{A}(\mathcal{O})$ over all open bounded $\mathcal{O} \subset \mathbb{R}^2$. Hence, σ_t is an automorphism on \mathcal{A} and implements the time evolution associated with the interacting field. Similarly, “locally correct” generators for the Lorentz boosts and the spatial translations can be defined, resulting in an automorphic action α on \mathcal{A} of the entire (identity component of the) Poincaré group \mathcal{P}_+^\uparrow in two spacetime dimensions.

For each $A \in \mathcal{A}$, we set $\omega_g(A) = \langle \Omega(g), A\Omega(g) \rangle$ ($\langle \cdot, \cdot \rangle$ denotes here the inner product on \mathcal{H}) to define the locally correct vacuum state ω_g of the interacting field. Taking a limit as the cutoff function g approaches the constant function 1, Glimm and Jaffe showed that $\omega_g(A) \rightarrow \omega(A)$, for each $A \in \mathcal{A}$, defines a new (locally normal) state ω on \mathcal{A} which is Poincaré invariant, *i.e.* $\omega(\alpha_{(\Lambda, x)}(A)) = \omega(A)$ for all $(\Lambda, x) \in \mathcal{P}_+^\uparrow$ and all $A \in \mathcal{A}$. Employing the GNS construction, one then obtains a new Hilbert space \mathcal{H} , a representation ρ of \mathcal{A} as a C^* -algebra acting on \mathcal{H} , and a vector $\Omega \in \mathcal{H}$ such that $\rho(\mathcal{A})\Omega$ is dense in \mathcal{H} and

$$\omega(A) = \langle \Omega, \rho(A)\Omega \rangle, \text{ for all } A \in \mathcal{A}.$$

In addition, one obtains a strongly continuous unitary representation U of the Poincaré group in two spacetime dimensions under which the algebras $\rho(\mathcal{A}(\mathcal{O}))$ transform covariantly, *i.e.*

$$U((\Lambda, x)) \rho(\mathcal{A}(\mathcal{O})) U((\Lambda, x))^{-1} = \rho(\mathcal{A}(\Lambda\mathcal{O} + x)).$$

Both the HAK and Wightman axioms have been verified for this model.

The generators of the strongly continuous Abelian unitary groups $\{\rho(e^{it\phi(f)}) \mid t \in \mathbb{R}\}$ and $\{\rho(e^{it\pi(f)}) \mid t \in \mathbb{R}\}$ satisfy the CCR. However, this representation of the CCR in \mathcal{H} is not unitarily equivalent to the initial representation in Fock space, in accordance with Haag’s Theorem. Indeed, by taking different values of the coupling constant λ in the above construction, one obtains an uncountably infinite family of mutually inequivalent representations of the CCR (see [70]).

It is in this representation (ρ, \mathcal{H}) that the field equations for this model find a mathematically satisfactory interpretation, as shown by Schrader [166]. And it is to the physically significant quantities in this representation that the corresponding perturbation series in λ is asymptotic — see below for further discussion. For this and other reasons, ω is interpreted as the exact vacuum state in the interacting theory corresponding to the Lagrangian interaction $\lambda\phi^4$, and the folium of states associated with this representation contains the physically admissible states of the interacting theory. Many further properties of physical relevance have been proven for this model more recently — see the discussion below in Section 3.1.

The results attained for the ϕ_2^4 model were subsequently extended to $P(\phi)_2$ models (using a periodic box cutoff), where $P(\phi)$ is any polynomial bounded from

¹²One can then show that the algebra $\mathcal{A}(\mathcal{O})$ coincides with the von Neumann algebra generated by bounded functions of the self-adjoint field operators $\int \phi(t, x)f(t, x) dx dt$, with test functions $f(t, x)$ having support in \mathcal{O} .

below [86–88].¹³ (See [88, 90] for more complete references and history of this development.) Hoegh-Krohn employed the techniques of Glimm and Jaffe to construct models in two spacetime dimensions (with similar results) in which the polynomial interaction $P(\phi)$ is replaced by a function of exponential type, the simplest example being $e^{\alpha\phi}$ [107].

Analogous results were proven for Y_2 , the Yukawa model in two spacetime dimensions by Glimm and Jaffe and Schrader [85, 89, 165]. In this model one commences with the direct product $\mathcal{H}_0 = \mathcal{H}_b \otimes \mathcal{H}_f$ of the Fock space \mathcal{H}_b for a scalar hermitian Bose field $\phi(t, x)$ of mass $m_b > 0$ and the Fock space \mathcal{H}_f for a Fermi field $\psi(t, x)$ of mass $m_f > 0$. In this model the free Hamiltonian H_0 is the total energy operator of the free fields ϕ and ψ . Because there is still an ultraviolet divergence remaining after Wick ordering, the cutoff interacting Hamiltonian is $H(g, \kappa) \equiv H_0 + H_I(g, \kappa) + c(g, \kappa)$, where $H_I(g, \kappa)$ is the result of applying a certain multiplicative ultraviolet cutoff (which is removed in the limit $\kappa \rightarrow \infty$) to the formal expression

$$H_I(g) \equiv \lambda \int g(x) \phi(0, x) : \bar{\psi}\psi : (0, x) dx,$$

and $c(g, \kappa)$ is a (finite) renormalization counterterm determined by second-order perturbation theory which diverges as $\kappa \rightarrow \infty$ [85] and includes both a mass and vacuum energy renormalization. With both volume and ultraviolet cutoffs in place, $H(g, \kappa)$ is a well defined operator on \mathcal{H}_0 . Glimm and Jaffe show that as $\kappa \rightarrow \infty$ the operator $H(g, \kappa)$ converges in the sense of graphs to a positive self-adjoint operator $H(g)$ with an eigenvector $\Omega(g) \in \mathcal{H}_0$ of lowest energy 0. Once again, they control the limit as $g \rightarrow 1$ of the expectations $\omega_g(A)$ for all $A \in \mathcal{A}_b \otimes \mathcal{A}_f$ and obtain a state ω on $\mathcal{A}_b \otimes \mathcal{A}_f$ that provides a corresponding (GNS) representation of the fully interacting theory. Glimm and Jaffe [89] also proved that the Yukawa field equations are satisfied by the fields in that representation. A similar argument was applied to the $Y_2 + P(\phi)_2$ model by Schrader [164]), where

$$H_I(g) \equiv \lambda \int g(x) (\phi(0, x) : \bar{\psi}\psi : (0, x) + : P(\phi) : (0, x)) dx$$

and $P(\phi)$ is any polynomial bounded from below. The axioms of HAK and Wightman were shown to hold in these models (see [182]), at least for all sufficiently small values of the coupling constant λ . In addition, by using a mixture of algebraic and Euclidean methods Summers [182] showed that the model manifests further properties of physical relevance, such as the funnel property (also known as the split property) and all assumptions of the Doplicher–Haag–Roberts superselection theory (*cf.* [9, 104]). Therefore the model also admits the physically expected Poincaré covariant, positive energy, charged representations associated with the generator of the global $U(1)$ gauge group of the model, which are mutually unitarily inequivalent.

¹³If $P(\phi)$ is not bounded from below, then the corresponding cutoff Hamiltonian $H(g)$ is not bounded from below and the resulting model is not stable.

Along the lines employed in the construction of the Yukawa model in two space-time dimensions, Glimm and Jaffe [91] also showed for the ϕ_3^4 model that the spatially cutoff Hamiltonian $H(g)$ is densely defined, symmetric and bounded below by a constant $E(g)$ proportional to the volume of the support of g . The renormalization constants in the Hamiltonian $H(g)$ are again given by perturbation theory and involve counterterms to the vacuum energy and the rest mass of a single particle. The proof was technically more challenging than that for Y_2 , even though the results were more limited to a significant extent. There was real motivation to find an alternative approach, as described in the next section.

However, before proceeding to the next section we mention the Federbush model, a model of self-interacting fermions in two spacetime dimensions. First proposed by Federbush [61], the Lagrangian of the model is

$$\sum_{s=\pm 1} \bar{\psi}_s (\not{\partial} - m(s)) \psi_s - 2\pi\lambda \epsilon_{\mu\nu} J_1^\mu J_{-1}^\nu,$$

where $\epsilon_{10} = -\epsilon_{01} = 1$, $\epsilon_{00} = \epsilon_{11} = 0$, $J_s^\mu = \bar{\psi}_s \gamma^\mu \psi_s$ and $m(s) > 0$, $s = \pm 1$. Without cutoffs of any kind, a concrete realization of the Federbush model can be given in terms of certain exponential expressions on a suitable Fock space, and Ruijsenaars [159] proved that this realization satisfies the Wightman axioms when $\lambda \in (-\frac{1}{2}, \frac{1}{2})$ (Einstein causality is actually verified only for sufficiently small λ). Of particular interest, he proved that the associated Haag–Ruelle scattering theory is asymptotically complete. The S–matrix is nontrivial, but there is no particle production [160]. The Federbush model was the first non-superrenormalizable model for which any of these properties have been proven.

3 Functional Integral Constructions — Euclidean

The technical difficulties of the approach described in the preceding section were formidable, and in the early 1970’s a technically more manageable program replaced (at least temporarily) the earlier approach.¹⁴ In this newer program a (scalar Bose) Euclidean quantum field is a random variable valued distribution $\phi(x)$ with probability measure μ on the corresponding distribution space. The moments of this measure

$$S_n(x_1, x_2, \dots, x_n) \equiv \int \phi(x_1) \phi(x_2) \cdots \phi(x_n) d\mu(\phi), \quad n \in \mathbb{N},$$

are called the *Schwinger functions*. Formally, the connection between the Wightman functions and the Schwinger functions is described as follows: Let $x = (x_0, x_1, \dots, x_{d-1})$ be the coordinates of a point in d dimensional Minkowski space and consider $x^E = (-ix_0, x_1, \dots, x_{d-1})$. Then

$$S_n(x_1, x_2, \dots, x_n) = W_n(x_1^E, x_2^E, \dots, x_n^E), \quad n \in \mathbb{N}.$$

¹⁴See [95, 174, 187, 188] for some history and references.

In point of fact, it is known that the Wightman functions of a Wightman theory can be analytically continued to such Euclidean points $(x_1^E, x_2^E, \dots, x_n^E)$ [120, 187] so the above relation is meaningful in that sense. Under certain additional conditions the resulting Schwinger functions are the moments of a Euclidean invariant probability measure μ . For example, the Schwinger functions of the free scalar Bose field of mass $m_0 > 0$ on two dimensional Minkowski space are the moments of the Gaussian probability measure μ_C on $\mathcal{S}'(\mathbb{R}^2)$ with mean 0 and covariance $C = (-\Delta + m_0^2)^{-1}$. The corresponding two-point Schwinger function is

$$S_2(x - y) = (2\pi)^{-2} \int \frac{e^{ik \cdot (x-y)}}{k^2 + m_0^2} dk^2,$$

where all inner products are Euclidean.

Hence, from a Wightman field theory one can obtain in this manner a Euclidean field theory. Under certain circumstances one can also obtain a Wightman field theory from a Euclidean field theory by appropriate analytic continuation of Schwinger functions to obtain Wightman functions satisfying the Wightman axioms. Of the various sets of conditions known to permit this reconstruction of the real time theory from the imaginary time theory, we mention only the *Osterwalder–Schrader axioms* [152]. Here one commences with a sequence $\{S_n(x_1, x_2, \dots, x_n)\}_{n \in \mathbb{N}}$ of tempered distributions (again called Schwinger functions) completely symmetric in their arguments which are invariant under the natural action of the Euclidean group. One requires additionally of these distributions that they satisfy a certain cluster property (ergodicity under the “time” translations), the Osterwalder–Schrader positivity condition (also called reflection positivity) and a technical linear growth condition. Osterwalder and Schrader showed that an inductive construction of analytic continuations applied to these Schwinger functions yields a set of Wightman functions satisfying the Wightman axioms. They also proved that analytically continuing the Wightman functions of a Wightman theory in the manner indicated above results in Schwinger functions satisfying all of the Osterwalder–Schrader axioms except possibly the linear growth condition. Since the linear growth condition may fail, the Osterwalder–Schrader axioms may not be equivalent to the Wightman axioms.¹⁵

In general, a given sequence of Schwinger functions satisfying the Osterwalder–Schrader axioms need not be the moments of a measure μ . However, the general strategy of the Euclidean construction program was to construct a suitable probability measure μ on $\mathcal{S}'(\mathbb{R}^d)$ such that its moments satisfy the Osterwalder–Schrader axioms. Application of the Osterwalder–Schrader reconstruction theorem would then result in a corresponding relativistic quantum field model satisfying the Wightman axioms. One would then further study these models for properties of physical relevance that go beyond the basic axioms. This approach also stimulated a fruitful exchange of techniques and tools between quantum field theory on

¹⁵Finding a physically meaningful set of conditions on Schwinger functions which is equivalent to the Wightman axioms is an open problem — see [163] for a recent overview of the topic “equivalence of Wightman functions and Schwinger functions” with references.

one side and probability theory and classical statistical mechanics on the other.

The connections between Euclidean QFT and probability theory are even richer than what has been suggested above. Nelson [146] pointed out that the free scalar massive Bose Euclidean field is a Markov process, as is the finite volume $P(\phi)_2$ field. This line of thought led to the use of the stochastic processes which are solutions of suitable stochastic differential equations to construct models of quantum fields — see Section 5.

3.1 $P(\phi)_2$ models

The $P(\phi)_2$ models were reexamined from the point of view of the Euclidean construction program, and new results of physical relevance were obtained. In analogy to the real time construction sketched in Section 2, one commences with a local perturbation of the free field, here represented by the Gaussian measure μ_C . Once again, let $P(\phi)$ be a polynomial bounded from below and Λ be a rectangular region in Euclidean \mathbb{R}^2 . The Wick ordered powers of the Euclidean free field may be defined recursively by the conditions : $\phi^0 := 1$, $\partial : \phi^n : / \partial \phi = n : \phi^{n-1} :$ and $\int : \phi^n : d\mu_C = 0$, $n \in \mathbb{N}$. Let $U_\Lambda \equiv \int_\Lambda : P(\phi) : (x) d^2x$. Then $e^{-U_\Lambda} \in L^p(d\mu_C)$ for all $p < \infty$ [174], so

$$\mu_\Lambda \equiv \frac{e^{-\lambda U_\Lambda} \mu_C}{\int e^{-\lambda U_\Lambda} d\mu_C} \quad (3.1)$$

is a well defined probability measure on $\mathcal{S}'(\mathbb{R}^2)$, where the coupling constant $\lambda \geq 0$. One may then consider the corresponding Schwinger functions

$$S_n^\Lambda(x_1, x_2, \dots, x_n) \equiv \int \phi(x_1) \phi(x_2) \cdots \phi(x_n) d\mu_\Lambda, \quad n \in \mathbb{N}.$$

By employing cluster expansions Glimm, Jaffe and Spencer proved that the limits

$$S_n(x_1, x_2, \dots, x_n) \equiv \lim_{\Lambda \nearrow \mathbb{R}^2} S_n^\Lambda(x_1, x_2, \dots, x_n), \quad n \in \mathbb{N}$$

exist¹⁶ and satisfy the Osterwalder–Schrader axioms for all sufficiently small λ/m_0^2 [92]. Hence, one obtains a corresponding relativistic quantum field model.

Similar results can also be obtained, at least for a large class of polynomials, by commencing with a lattice approximation and controlling the limit as the lattice spacing goes to 0 [95, 174] and/or by using correlation inequalities instead of cluster expansions [95, 174]. In the former, placing the quantum field model on a lattice results in a classical Ising ferromagnetic system with unbounded spins and nearest neighbor coupling arising from the finite difference approximation to $-\Delta + m_0^2$. The lattice spacing serves as an ultraviolet cutoff. An example of the latter are the GKS-inequalities, which were extended from lattice ferromagnetic systems to

¹⁶In point of fact, the measures μ_Λ converge weakly to a measure μ_∞ whose moments are $S_n(x_1, x_2, \dots, x_n)$, $n \in \mathbb{N}$.

$P(\phi)_2$ models by Guerra, Rosen and Simon [102]: For $P(\phi) = Q(\phi) - h\phi$, where Q is an even polynomial and $h \geq 0$, one has

$$S_n^\Lambda(x_1, x_2, \dots, x_n) \geq 0, \quad n \in \mathbb{N},$$

and

$$S_{n+m}^\Lambda(x_1, x_2, \dots, x_{n+m}) \geq S_n^\Lambda(x_1, x_2, \dots, x_n) S_m^\Lambda(x_{n+1}, x_{n+2}, \dots, x_{n+m}), \quad n, m \in \mathbb{N},$$

in the sense of distributions. Nelson proved that the Schwinger functions are monotone in the volume cutoff Λ ,¹⁷ namely for suitably regular regions $\Lambda' \subset \Lambda$ one has

$$S_n^{\Lambda'}(f_1, f_2, \dots, f_n) \leq S_n^\Lambda(f_1, f_2, \dots, f_n)$$

for all positive test functions f_1, \dots, f_n and all $n \in \mathbb{N}$ [146]. Bounds on the Schwinger functions which are uniform in Λ yield the existence of the infinite volume limit. The GKS inequalities are used in the proof of both the monotonicity and the uniform bounds. Further arguments yield the validity of the Osterwalder–Schrader axioms for the limit Schwinger functions. Though use of correlation inequalities restricts the interaction polynomial, wherever correlation inequalities can be used, there is no restriction on the size of $\lambda \geq 0$. This is an advantage with respect to the results attained using cluster expansions.

In the weak coupling limit (sufficiently small λ/m_0^2) the “uniqueness of the vacuum” follows from exponential clustering bounds on the Schwinger functions (or the Wightman functions): there exists an $m > 0$ such that for all $0 \leq k \leq n \in \mathbb{N}$ and $a \in \mathbb{R}^2$

$$S_n(x_1, x_2, \dots, x_k, x_{k+1} - a, x_{k+2} - a, \dots, x_n - a) = \quad (3.2)$$

$$S_k(x_1, x_2, \dots, x_k) S_{n-k}(x_{k+1}, x_{k+2}, \dots, x_n) \leq C e^{-m\|a\|}. \quad (3.3)$$

In fact, the bound (3.2) establishes also the existence of a mass gap at least as large as m , *i.e.* the spectrum of the mass operator is contained in $\{0\} \cup [m, \infty)$. Indeed, it was shown that there exists an $m > 0$ so that the spectrum of the mass operator is contained in $\{0, m\} \cup [m_1, \infty)$ and that as $\lambda \rightarrow 0$ with m_0 fixed, one has $m \rightarrow m_0$ and $m_1 \rightarrow 2m_0$ [92]. Hence, weakly coupled $P(\phi)_2$ models have an isolated one-particle hyperboloid, and the Haag–Ruelle scattering theory (see [9, 120]) may be applied [92], resulting in a well defined scattering theory for these models. It has been shown by Osterwalder and S  n  or [153] that the resultant S–matrix is nontrivial and that the usual perturbation expansion in the coupling constant for this S–matrix is asymptotic to the exact S–matrix. In these models there is particle production.

Much work has been expended in the study of the particle structure of $P(\phi)_2$ models which goes beyond the existence of an isolated mass hyperboloid, since this is of direct physical interest and also because it appears likely that such knowledge would be a necessary prerequisite for any proof of the asymptotic completeness

¹⁷Strictly speaking, “half-Dirichlet boundary conditions” on Λ are introduced to prove this.

of the models. An early example is the proof by Spencer and Zirilli [177] that in the $\lambda\phi_2^4$ model there are no even bound states of energy less than $4(m - \epsilon)$, where $m = m(\lambda)$ is the physical mass of the model and $\epsilon \rightarrow 0$ as $\lambda \rightarrow 0$. On the other hand, Dimock and Eckmann [45] showed that in the $\lambda(\phi^6 - \phi^4)_2$ model there exists a unique two particle bound state for sufficiently small λ , and it has mass $2m(1 - \frac{9}{8m^4}\lambda^2 + O(\lambda^3))$. They subsequently generalized these results [46] to any polynomial interaction having no positive ϕ^4 term. They further showed that if there is such a ϕ^4 term, then there is no two particle bound state. In both cases, there are no bound states masses embedded in the continuum below $3m - \epsilon$, *i.e.* the model is two-particle asymptotically complete. Neves da Silva [147] proved that when the interaction polynomial is even and there exists a two particle bound state, then there also exists a three particle bound state of energy less than $3m$. The uniqueness of the former entails the uniqueness of the latter. Further details of particle structure are strongly model dependent, but general results attained by Bros and Iagolnitzer [27] are a useful starting point.

Using Euclidean Markov fields and correlation inequalities, Albeverio and Hoegh-Krohn [2] proved that the infinite volume limit of scalar bosons with even exponential self-interaction in two space-time dimensions satisfies the Osterwalder–Schrader axioms, including clustering, so that the corresponding Minkowski space theory satisfies the Wightman axioms, including uniqueness of the vacuum. Together with Gallavotti they showed [3] that when $d \geq 3$ the model can also be constructed, but its Schwinger functions coincide with those of the corresponding free field, *i.e.* it is trivial.

In general QFT the “uniqueness of the vacuum” can fail in a physically significant manner — the quantum field model can manifest “phase transitions” entirely analogous to those found in statistical mechanics. In this circumstance the subspace \mathcal{H}_0 of vectors in \mathcal{H} invariant under the action of $U(\mathcal{P}_+^\dagger)$ is not one-dimensional, and one can find two unit vectors $\Omega_1, \Omega_2 \in \mathcal{H}_0$ and an observable A (often called the order parameter) such that $\langle \Omega_1, A\Omega_1 \rangle \neq \langle \Omega_2, A\Omega_2 \rangle$. As in statistical mechanics, one can determine “phase diagrams” plotted against the various parameters of interaction in the model with “critical points” where phase transition curves end.

The first such results were proven for $P(\phi) = (\phi^2 - \sigma^2)^2/\sigma^2$ for σ a sufficiently large constant. By a suitable change of variables and Wick ordering, this is physically equivalent to the choice $P(\phi) = \lambda\phi^4 + \frac{1}{2}m_0^2\phi^2$ for sufficiently small m_0^2/λ .¹⁸ As in statistical mechanics, one displays the phase transition by considering the interaction $P(\phi) = (\phi^2 - \sigma^2)^2/\sigma^2 - h\phi$, with h a constant representing an “external field”. For $h \neq 0$ the corresponding model has a unique vacuum, by the Lee-Yang theorem generalized to the ϕ_2^4 model by Simon and Griffiths [173], *i.e.* the corresponding measure $\mu_{\sigma,h}$ is ergodic. For all sufficiently large σ , Glimm, Jaffe and

¹⁸In direct analogy to statistical mechanics, the limit $m_0^2/\lambda \rightarrow 0$ is referred to as the low temperature limit and that of $\lambda/m_0^2 \rightarrow 0$ is called the high temperature limit.

Spencer show that

$$\lim_{h \searrow 0} \langle \phi(f) \rangle_{\sigma, h} = - \lim_{h \nearrow 0} \langle \phi(f) \rangle_{\sigma, h} > 0,$$

for positive test function f , where $\langle \cdot \rangle_{\sigma, h}$ is the expectation with respect to $\mu_{\sigma, h}$. Hence, there is a phase transition at $h = 0$ for sufficiently large σ and the field itself is an order parameter. Indeed, Glimm, Jaffe and Spencer [94] subsequently modified the cluster expansion to show that with $P(\phi) = \lambda\phi^4 - \frac{1}{4}\phi^2 - h\phi - E_c$ (E_c chosen so that the infimum of the polynomial is 0) and μ_b the Gaussian measure on $\mathcal{S}'(\mathbb{R}^2)$ with mean $b = \pm(8\lambda)^{-1/2}$ (note these are the minima of $P(\phi)$) and covariance $C = (-\Delta + 1)^{-1}$, then if h and b have the same sign the Schwinger functions associated with the measure

$$\mu_\Lambda \equiv \frac{e^{-\int_\Lambda (P(\phi)(x) + \frac{1}{2}(\phi-b)^2(x))d^2x} d\mu_b}{\int e^{-\int_\Lambda (P(\phi)(x) + \frac{1}{2}(\phi-b)^2(x))d^2x} d\mu_b} \quad (3.4)$$

converge as $\Lambda \nearrow \mathbb{R}^2$ to Schwinger functions satisfying the Osterwalder–Schrader axioms for all sufficiently small λ , in particular, the corresponding vacuum is unique and the limit measure μ_\pm is ergodic. Note that in (3.4) the mean and mass in the Gaussian measure exactly cancel the term added to $P(\phi)$ in the region Λ , leaving a \pm boundary condition outside of Λ . What is more, $\langle \phi(x) \rangle_\pm = b + O(\lambda^{3/2})$, where $\langle \cdot \rangle_\pm$ is the expectation in μ_\pm . Hence, for $h = 0$ one obtains two distinct Wightman theories in which the corresponding vacuum expectations of the field differ by a sign and in which the $\phi \rightarrow -\phi$ symmetry of the action is spontaneously broken. In analogy to situations arising in statistical mechanics, one says that at $h = 0$ there are two distinct “phases” for the model. Koch [129] showed that in each of these pure phases the mass hyperboloid is isolated, so that Haag–Ruelle scattering theory may be applied, and that there exists a two particle bound state.

The mean field cluster expansion of Glimm, Jaffe and Spencer was subsequently extended by Summers [181] to an interaction involving a sixth degree polynomial, which resulted in a value of the interaction coefficients at which three distinct phases coexist and in the corresponding phase diagram there are phase lines which are not associated with spontaneous symmetry breaking. There are also values of the coefficients resulting in two “critical points” where phase lines simply end, *i.e.* where the distinction between the phases vanishes. Each of the pure phases satisfies all of the Osterwalder–Schrader axioms, and the perturbation series for the Schwinger functions (suitably adjusted about the mean value of the field in the phase) is asymptotic to the exact (similarly adjusted) Schwinger functions. Imbrie [113] generalized this study of phase transitions in $P(\phi)_2$ models to very general polynomials yielding quite complex phase diagrams by suitably adapting techniques developed in statistical mechanics to this situation.

Although most effort has been expended on the construction of quantum field models in a vacuum representation, other representations are of physical interest, as well. One such class of representations studied by constructive and axiomatic quantum field theorists is the class of equilibrium thermal representations, *i.e.*

representations associated with states which satisfy the KMS condition for some fixed inverse temperature (*cf.* [26] for further background and references). Here we only mention results proven in concrete interacting models. Hoegh-Krohn [108] considered scalar Bose fields in two spacetime dimensions with either polynomial or exponential self-interaction and showed using a hybrid of algebraic and Euclidean techniques that for any $T > 0$ the infinite volume limit of the spacetime cutoff Gibbs state at temperature T exists on a suitable global algebra of observables and satisfies both the KMS condition for $\beta = T^{-1}$ and the cluster property and is translation invariant. More recently, Gérard and Jäkel [77] revisited the matter and provided a different proof of these results, as well as some further, more technical observations.

Thermal equilibrium states cannot be Lorentz invariant [150], even if the equations of motion are invariant under Poincaré transformations and the signal propagation speed is finite. Nonetheless, as recognized by Bros and Buchholz [28], the *passivity* (*cf.* [26]) of an equilibrium state should still be visible to an observer in motion with respect to the rest frame distinguished by the KMS state. They showed that this entails that the thermal equilibrium states of a relativistic QFT should have stronger analyticity properties in configuration space than those already implicit in the KMS condition. They formulated these properties as a *relativistic KMS condition*. Using the Euclidean approach to thermal fields and technical advances in treating spatially cutoff models due to Klein and Landau [127], Gérard and Jäkel [78] proved that the two-point function in the $P(\phi)_2$ model satisfies this relativistic KMS condition, and Jäkel and Robl [115] extended the result to general n -point functions in this model.

Other non-vacuum representations of physical interest which have been studied rigorously in concrete models are *charged* representations, an example of which is provided by the Yukawa₂ model [182], where the countably infinite eigenspaces of a global “charge” operator Q commuting with all observables and the representation of the Poincaré group yield mutually inequivalent representations of the observables (“superselection sectors”) that satisfy the HAK axioms, excepting the existence of the vacuum (the vacuum sector is the charge 0 eigenspace of Q).

Soliton representations can arise with the appearance of phase transitions and associated spontaneous breaking of symmetries of the model, as in the ϕ_2^4 models discussed above. They are believed to occur primarily in models in two spacetime dimensions, though there are heuristic indications that they might exist in non-abelian Yang–Mills models in four spacetime dimensions. They are Poincaré covariant, positive energy representations of the fields (observables) that, in a certain sense, interpolate between two distinct vacuum representations. As an illustration, consider the ϕ_2^4 model in the parameter range where the phase transition discussed above occurs. At $\hbar = 0$ there exist two vacuum vectors Ω_{\pm} inducing two vacuum states $\omega_{\pm}(\cdot) \equiv \langle \Omega_{\pm}, \cdot \Omega_{\pm} \rangle$ on the fields, equivalently on the algebra of observables \mathcal{A} . As shown by Fröhlich [71], there exists a localized automorphism σ on \mathcal{A} such that $\omega_{\pm} \circ \sigma$ are states on \mathcal{A} and the GNS representations of \mathcal{A} corresponding to $\omega_{\pm} \circ \sigma$ are Poincaré covariant and satisfy the spectrum condition. If \mathcal{H}_{\pm} are the

representation spaces of \mathcal{A} corresponding to ω_{\pm} and $\mathcal{H}_{s,s'}$ the representation spaces corresponding to $\omega_{\pm} \circ \sigma$, then $\mathcal{H}_+ \oplus \mathcal{H}_- \oplus \mathcal{H}_s \oplus \mathcal{H}_{s'}$ is the full Hilbert space of the ϕ_2^4 model (in the stated parameter range). On this space a time-independent “topological” charge operator is defined formally by

$$Q = \int \frac{\partial}{\partial x} \phi(t, x) dx.$$

$\mathcal{H}_+ \oplus \mathcal{H}_-$ is the eigenspace of Q corresponding to the eigenvalue 0, and \mathcal{H}_s , resp. $\mathcal{H}_{s'}$, is the eigenspace corresponding to the eigenvalue $2\langle \Omega_+, \phi(x)\Omega_+ \rangle > 0$, resp. $2\langle \Omega_-, \phi(x)\Omega_- \rangle = -2\langle \Omega_+, \phi(x)\Omega_+ \rangle$. The sense in which the soliton, resp. anti-soliton, state ω_s , resp. $\omega_{s'}$, interpolates between the two vacua is suggested by

$$\lim_{x \rightarrow \pm\infty} \omega_s(\phi(t, x)) = \omega_{\pm}(\phi(t, x))$$

and

$$\lim_{x \rightarrow \pm\infty} \omega_{s'}(\phi(t, x)) = \omega_{\mp}(\phi(t, x)).$$

Although not all details have been published, there are strong indications in the work of Fröhlich and Marchetti [71, 74] that there are single soliton (antisoliton) states in \mathcal{H}_s ($\mathcal{H}_{s'}$) which are created out of the vacuum vectors Ω_+ (Ω_-) by a local field $s(x)$, called the soliton field. B  llisard, Fr  hlich and Gidas [18] have proven that for small enough coupling λ the mass of the soliton grows like λ^{-1} .

These ideas have been explored also in $P(\phi)_2$ models, the Yukawa model and others. Of particular interest is the sine-Gordon model (see Section 3.6), which, because of the periodicity of the interaction, admits countably infinitely many soliton sectors [71]. These and other models have also been studied from the Euclidean point of view by Fr  hlich and Marchetti [74], casting a new perspective on these results. See also Schlingemann [162] for more recent developments.

3.2 The ϕ_3^4 model

Also the ϕ_3^4 model was revisited using Euclidean techniques, yielding stronger results. As seen in Section 2, the ϕ_3^4 model is superrenormalizable and requires an infinite mass renormalization. Hence, both a volume and an ultraviolet cutoff are necessary to begin with well defined quantities. Let μ_{C_κ} be the Gaussian measure on the Schwartz distribution space $\mathcal{S}'(\mathbb{R}^3)$ with mean 0 and covariance C_κ , with kernel

$$C_\kappa(x - y) = (2\pi)^{-3} \int \frac{e^{ik \cdot (x-y)}}{k^2 + m_0^2} \eta(k/\kappa) dk^3, \quad (3.5)$$

$m_0 > 0$, where η is a smooth function of compact support taking the value 1 in a neighborhood of the origin in \mathbb{R}^3 . The cutoff free field ϕ_κ is the corresponding random variable valued distribution. The cutoff action is

$$V(\lambda, \Lambda, \kappa) \equiv \int_{\Lambda} \lambda : \phi_\kappa^4 : (x) + c(\kappa, \lambda, \phi_\kappa(x)) d^3x,$$

where $c(\kappa, \lambda, \phi_\kappa(x))$ is again a renormalization counterterm suggested by perturbation theory (see *e.g.* [138]), and $\lambda > 0$ is the coupling constant. Define the cutoff interacting measure to be

$$\mu_{\lambda, \Lambda, \kappa} \equiv \frac{e^{-V(\lambda, \Lambda, \kappa)} \mu_{C_\kappa}}{\int e^{-V(\lambda, \Lambda, \kappa)} \mu_{C_\kappa}}.$$

Using phase cell and cluster expansions, Feldman and Osterwalder [63], on one hand, and Magnen and Sénéor [138], on the other, have independently shown that for sufficiently small λ/m_0^2 the corresponding Schwinger functions converge as $\kappa \rightarrow \infty$, $\Lambda \nearrow \mathbb{R}^3$, to Schwinger functions satisfying the Osterwalder–Schrader axioms, so that there is a corresponding Wightman theory. By using different techniques going back to Nelson and Guerra, Seiler and Simon [171] were able to remove the restriction on λ/m_0^2 . Using a lattice approximation and correlation inequalities, Park [155] attains the same results. The resulting quantum fields satisfy the corresponding ϕ^4 field equations [64], are locally associated with a net of algebras satisfying the HAK axioms [171], and the perturbation expansion for the Schwinger functions is Borel summable [140]. Burnap [38] showed the existence of one particle states in this model, so the Haag–Ruelle scattering theory is applicable to this model.

In the intervening thirty years the model has been revisited from many points of view; much effort has been exerted to simplify the techniques used in the original proofs (*cf.* the recent [143]), and we mention, in particular, that various rigorous versions of the renormalization group have been developed and applied to the ϕ_3^4 model (*cf.* [32] for an overview). This approach and a second, using self-avoiding random walks to construct ϕ_d^4 models for $d \geq 2$, are briefly discussed in Section 3.3.

3.3 ϕ_d^4 models and their dependence on d

According to the classification arising from the application of standard perturbation theory to Lagrangian-based quantum field models, the ϕ_d^4 model is superrenormalizable when $d = 2, 3$, renormalizable when $d = 4$, and nonrenormalizable when $d \geq 5$. As seen above, the ϕ_2^4 and ϕ_3^4 models have been successfully constructed with many properties of physical relevance verified. However, the study of ϕ_4^4 has made clear that there are further important subtleties to be understood in the construction of renormalizable quantum field models. Some of these are discussed in this section.

It is a widespread view that quantum field models which are not asymptotically free (see below), such as ϕ_4^4 with positive coupling constant, are not mathematically consistent. A close study of ϕ_4^4 shows that this view must be nuanced. Another, closely related point is that perturbation theory is the primary tool used by field theorists to make predictions that may be checked against laboratory measurements. However, it is widely believed that perturbation series in QFT are *divergent*, at least in models involving a bosonic field (this has been proven for $P(\phi)_2$

models by Jaffe [117] and for the ϕ_3^4 model by de Calan and Rivasseau [40], but the initial suspicion that this should be true goes back to a simple heuristic argument of Dyson [58] for QED). So one can hope that perturbation theory is *asymptotic* to an exact model, or even better, since the connection between the perturbation theory and the exact theory is then tighter as the quantities in the exact theory can be uniquely recovered by the Borel procedure from the corresponding perturbation series, the perturbation series is *Borel summable*. The Borel summability of the perturbation series for a number quantities of interest has been verified in many of the models constructed to this date. However, an examination of ϕ_4^4 is revealing also in this connection.¹⁹

Many authors have proven bounds which establish the local existence of the Borel transform for the standard perturbation series for ϕ_4^4 (*i.e.* the Borel transform of the series exists and is analytic in a disk centered at 0), the most recent of which is due to Kopper [131] (an overview and references to the earlier work may be found in [131], as well).

Important insights have been won in QFT through the development of *renormalization group* techniques. These have many concrete realizations, some of which have been established in a mathematically rigorous manner. We briefly describe one of the latter type in the specific case of ϕ_d^4 . Let C_κ be a suitable ultraviolet cutoff free covariance such as (3.5) or

$$C_\kappa = (-\Delta + m_0^2)^{-1} e^{-(\Delta + m_0^2)/\kappa^2},$$

for which Euclidean square momenta larger than κ^2 are either strongly suppressed or totally eliminated and for which $C_\kappa \rightarrow (-\Delta + m_0^2)^{-1}$ in a suitable sense as $\kappa \rightarrow \infty$. The corresponding cutoff Euclidean measure is

$$\mu_{\lambda, \Lambda, \kappa} \equiv \frac{e^{-V(\lambda, \Lambda, \kappa)} \mu_{C_\kappa}}{\int e^{-V(\lambda, \Lambda, \kappa)} \mu_{C_\kappa}},$$

where

$$V(\lambda, \Lambda, \kappa) \equiv \int_\Lambda \lambda_\kappa Z_\kappa^2 : \phi_\kappa^4 : (x) + \frac{1}{2} Z_\kappa \delta m_\kappa^2 : \phi_\kappa^2 : (x) d^d x. \quad (3.6)$$

This expression involves a wave function renormalization Z_κ , a mass renormalization δm_κ^2 and a coupling constant renormalization $\delta \lambda_\kappa = \lambda_\kappa - \lambda$, all of which are κ -dependent and some or all of which divergent as $\kappa \rightarrow \infty$, depending on the value of d . These quantities can be given explicitly and are motivated by standard perturbation theory, as above. As seen above, when $d = 2, 3$ Z_κ may be chosen to be 1 and the mass and coupling constant renormalizations are polynomials in λ . When $d = 4$ the three renormalization counterterms are given by power series in

¹⁹It should be emphasized that even when the series is Borel summable, at some order of the expansion the difference between the exact theory (if it exists) and the perturbation series prediction will get larger, not smaller, as one takes higher and higher orders into account. The same is true of the difference between the experimentally observed result and the perturbation series prediction.

λ which are likely to be divergent. They must therefore be defined implicitly, and for this purpose renormalization group techniques can be applied.

The basic idea of the renormalization group applied in this context is to break up any integral with respect to $\mu_{\lambda,\Lambda,\kappa}$ into a sequence of integrals each over the field restricted to a particular range of momenta. To outline one concrete realization of this idea, let $\delta C = C_\kappa - C_{\kappa'}$ with $\kappa' < \kappa$ and $\delta\phi = \phi_\kappa - \phi_{\kappa'}$. This split of the field into “high” and “low” momentum parts $\phi_\kappa = \phi_{\kappa'} + \delta\phi$ leads to the replacement of $\mu_{\lambda,\Lambda,\kappa}$ with a suitable product measure $\mu_{\lambda,\Lambda,\kappa'} \times \mu_{\lambda,\Lambda,\delta C}$. Carrying out the integration over the measure on the right hand results in an *effective action* $V_{\kappa'}(\phi_{\kappa'})$ determined by setting the value of the said integration equal to

$$e^{-V_{\kappa'}(\phi_{\kappa'})} \mu_{C_{\kappa'}}$$

up to a normalization factor. A computation determines the lower order in λ_κ contributions to $V_{\kappa'}$, and with a suitable choice of $Z_{\kappa'}$ one finds that $V_{\kappa'}$ has, up to “small” terms, the same form as (3.6).²⁰ The new renormalization terms $Z_{\kappa'}, \delta m_{\kappa'}^2, \lambda_{\kappa'}$ can be computed as functions of $Z_\kappa, \delta m_\kappa^2, \lambda_\kappa$, resulting in the renormalization group “flow equations”.²¹ In order that the “small” terms actually are small, it is necessary to choose κ' suitably close to κ and λ_κ suitably small. Thus, beginning with a large value of κ one must proceed in many incremental steps down from $\kappa_N = \kappa$ to a conveniently small κ_0 at which the final integration can be relatively easily estimated to provide the desired bounds. But since the limit $\kappa \rightarrow \infty$ must ultimately be controlled, one would also need to have $\lambda_{\kappa_N} \rightarrow 0$ as $\kappa_N = \kappa \rightarrow \infty$, *i.e.* as $N \rightarrow \infty$. If this is so, then the model is said to be *asymptotically free* (in the ultraviolet regime).

In ϕ_4^4 the flow equation for $\lambda_n \equiv \lambda_{2^n}$ is

$$\lambda_{n-1} \approx \lambda_n - \beta_2 \lambda_n^2 - \beta_3 \lambda_n^3$$

with $\beta_2 > 0$, so that for small λ_n the preceding λ_{n-1} is smaller than λ_n , contrary to the above picture. This is a signal that the $\lambda\phi_4^4$ theory is not asymptotically free and the renormalization group techniques cannot be applied in this case.

However, the same flow equation shows that for $\lambda < 0$ this procedure may be applicable. In fact, Gawedzki and Kupiainen [76] have carried out this procedure to provide a rigorous construction of the (hierarchical) Euclidean $\lambda\phi_4^4$ model for negative coupling constant. The standard renormalized perturbation expansion is asymptotic for the Schwinger functions. However, since reflection positivity is most likely not satisfied in this model, one does not arrive ultimately at a Minkowski space theory. Nonetheless, for spacetime dimensions less than 4 the corresponding

²⁰These “small terms” are generally ignored in heuristic QFT; however, showing that they can be controlled is one of the main technical problems in the rigorous use of renormalization group ideas.

²¹In heuristic QFT the flow equations are usually expressed in terms of differential equations, *e.g.* the Callan–Symanzik equation, which, however, have not been given a mathematically rigorous, nonperturbative basis, in general.

flow equation for ϕ_d^4 turns out to be consistent with the above picture also for *positive* coupling constant, and this approach yields another rigorous construction of the $\lambda\phi_3^4$ model for positive λ .

Another approach to the study of ϕ_d^4 models was motivated by Symanzik's insight [188] that Euclidean ϕ_d^4 theory can be understood as a classical gas of weakly self-avoiding random paths and loops. To get some idea of this, consider the operator $-\Delta + m_0^2$ in its approximation as a difference matrix on the lattice \mathbb{Z}^d , and write it as a sum of diagonal and off-diagonal terms:

$$\beta^{-1}I - J \equiv (2d + m_0^2)I - J,$$

where I is the identity matrix. The entries of the matrices I, J are indexed by the lattice sites in \mathbb{Z}^d , and the value of J_{xy} is 1 if xy is a lattice bond and 0 otherwise. So the covariance of the corresponding Gaussian lattice measure can be written as

$$(-\Delta + m_0^2)_{xy}^{-1} = (\beta^{-1}I - J)_{xy}^{-1} = \sum_{n=0}^{\infty} \beta^{n+1} (J^n)_{xy}.$$

The quantity $(J^n)_{xy}$ can be interpreted in this picture as a sum over all possible walks along the lattice which go from site y to site x in n steps. Observe that $\beta = (2d + m_0^2)^{-1} < 1$ and that J_{xy} is exponentially damped as the distance between x and y grows to infinity. If w is a path (a “walk”) along the lattice using nearest neighbor bonds in the lattice (here viewed as a (hyper)cubic lattice) and $|w|$ is the number of lattice bonds in the walk w , then the random walk representation of the covariance is

$$(-\Delta + m_0^2)_{xy}^{-1} = \sum_{w: y \rightarrow x} \beta^{|w|+1} (J^{|w|})_{xy},$$

where the sum is over all walks from y to x . Symanzik showed how one could use this random walk representation for the two-point Schwinger function of the free field to give an expression for the Schwinger functions of the interacting model (on the lattice, in finite volume). This results in studying correlations of random paths, where the weight functions on the random paths is significantly more complicated than the straightforward exponential weight above [188]. A multitude of different random walk representations is to be found in the literature - see the monograph by Fernandez, Fröhlich and Sokal [67] for a unified presentation of many of these.

Typically, the random walk formalism is used in a finite volume, lattice approximation of the model to derive various kinds of correlation inequalities, which then generally carry over easily to the infinite volume limit. These correlation inequalities are utilized to establish bounds on the continuum limit, *i.e.* the exact model, and on the critical exponents of the model. Using these techniques, it has been definitively established by Aizenman [1] and Fröhlich [72] that for $d > 4$ the exact limit theory is trivial, *e.g.* the exact model is Gaussian, manifest no interaction, has trivial scattering. On the other hand, in the case of $d = 4$ the situation is more subtle, as the limit may depend on the way the limits are taken and which details enter into the particular mode of construction — the reader is referred to [67] for

details. In many of these circumstances it has been proven that the exact theory is again trivial in four spacetime dimensions. This is striking, since the standard renormalized perturbation series for the model exists to all orders and is not trivial. Hence, the perturbation series is *not* asymptotic to what is apparently the exact theory. Moreover, the classical limit of that exact quantum theory could not coincide with the classical ϕ_4^4 field theory.

This brings us to the question: which criteria do we use to decide when a given mathematical model is a/the M_d quantum field theory? For most of the models discussed to this point one has the reassurance that the standard (possibly renormalized) perturbation series is asymptotic to some set of important quantities, *e.g.* the Schwinger functions, in the “exact model” (and in many cases the latter quantities can be uniquely recovered from the Borel transform of the former series). For some of these theories one has even been able to prove that in the “exact model” the fields satisfy the (suitably interpreted) semiclassical motivated field equations. For further reassurance, some researchers would also like to know that (A) the classical limit of the “exact model” coincides with (B) the classical theory associated with the Lagrangian selected by whatever semiclassical reasoning went into the choice of the model. This latter question has not yet received much attention from mathematical physicists, but Donald [49] has established a result for $P(\phi)_2$ models with convex polynomial P which relates the classical limit of certain quantities in the quantum field model to corresponding quantities in the classical model. However, the relation between (A) and (B) is not a 1-1 correspondence, even in the well-behaved $P(\phi)_2$ models. For example, Slade [175] has shown that for values a of the classical field for which the classical potential $U_0(a) = P(a) + \frac{1}{2}m^2a^2$ does not equal its convex hull, the correspondence between (A) and (B) can break down.

As already seen, none of these desiderata is satisfied by the (trivial) “exact model” for ϕ_4^4 . These and other considerations have motivated Klauder to propose an alternative way to construct “exact” ϕ_4^4 . The basic idea is not to perturb the free measure, introduce counterterms motivated by standard perturbation theory and then control a number of limits, as has been done in the work indicated above, but to perturb a “pseudofree” measure, introduce a different set of counterterms and then control certain limits to get the “exact model” [126]. This program has not been completed, so the reader is referred to [125, 126] for more details.

3.4 Yukawa $_d$ models

Theories with fermions are amenable to Euclidean methods, as well, if the interactions are quadratic in the fermions [151], including therefore Yukawa-type interactions, *i.e.* interactions of the form $\bar{\psi}\Gamma\psi\phi$, $\Gamma = 1, i\gamma_5 = -\gamma_0\gamma_1$, for the scalar, respectively pseudoscalar, coupling. Initially, the Fermi fields were “integrated out”, following Matthews and Salam, resulting in an effective action involving only the boson field called the Matthews-Salam determinant. Due to the necessity for a mass renormalization, the renormalized Matthews-Salam determinant is

given by

$$\det_{\text{ren}}(1 + \lambda K(\phi)) \equiv \det_3(1 + \lambda K(\phi)) \exp \left[\frac{\lambda^3}{3} \text{tr} K^3(\phi) + \delta m^2 \int : \phi^2 : (x) d^2 x \right],$$

where

$$\det_n(1 + A) \equiv \det \left[(1 + A) e^{\sum_{k=1}^n \frac{1}{k} (-A)^k} \right],$$

$K = S\Gamma\phi$, S is the Euclidean fermion propagator and δm^2 is a boson mass counterterm arising from second order perturbation theory. The ultraviolet cutoff index κ has been suppressed. Seiler [170] showed that, after introducing a suitable finite volume cutoff into K and $\int : \phi^2 : (x) d^2 x$, the renormalized determinant has the necessary integrability properties with respect to the free boson measure μ_C as $\kappa \rightarrow \infty$. Subsequently, Magnen and Sénéor [139] and Cooper and Rosen [43] independently verified the Osterwalder–Schrader axioms for the Schwinger functions of the infinite volume limit by using cluster expansions. The Fermi fields in the Schwinger functions are also “integrated out”: the Schwinger function for n boson fields, m fermion and m anti-fermion fields is given by

$$Z^{-1} \int \left(\prod_{l=1}^n \phi(x_l) \right) \det [S'(y_i, z_k; \phi)] \det_{\text{ren}}(1 + \lambda K(\phi)) d\mu_C(\phi), \quad (3.7)$$

where

$$Z \equiv \int \det_{\text{ren}}(1 + \lambda K(\phi)) d\mu_C(\phi),$$

the determinant is applied to the matrix whose (i, j) th entry is $S'(y_i, z_k; \phi)$, $i, j = 1, \dots, m$, which is the two point Schwinger function for the fermions in the external field ϕ and determined by $(1 + \lambda K)S' = S$ (all cutoffs are suppressed for transparency).

Renouard [157] proved that the perturbation expansion for the Schwinger functions of the limit theory is Borel summable. Balaban and Gawedzki [13] established the existence of a phase transition in the two-dimensional Euclidean pseudoscalar Yukawa model for large fermion mass. To do so, they adapted the mean field cluster expansion method developed by Glimm, Jaffe and Spencer [94] to this situation.

Lesniewski [137] utilized advances in CQFT developed for purely fermionic models to reverse the initial procedure. He “integrated out” the bosonic field first, resulting in an effective fermionic action of the form

$$\frac{1}{2} \int g_\kappa(x - y) : \bar{\psi}\psi : (x) : \bar{\psi}\psi : (y) dx dy,$$

where

$$g_\kappa(x - y) \equiv \frac{\lambda^2}{(2\pi)^2} \int \frac{e^{ip(x-y)}}{p^2 + m^2 + \delta m_\kappa^2(\lambda)}.$$

λ is the coupling constant and the mass counterterm $\delta m_\kappa^2(\lambda)$ is given by second order perturbation theory. Using renormalization group techniques developed to

treat the Gross-Neveu model (see the next section), he reproved the existence and the Borel summability of the limit theory.

By modifying the phase space cell expansion developed by Glimm and Jaffe to deal with the ϕ_3^4 model, Magnen and Sénéor [141] established the Osterwalder–Schrader axioms and the Borel summability of the standard perturbation theory for the Schwinger functions of the pseudoscalar Yukawa₃ model and also indicated how to prove the corresponding results for the scalar Yukawa₃ model.

3.5 Gross–Neveu₂ model

The Lagrangian for the (massive) Gross–Neveu model is given by

$$\bar{\psi}(x) (i\zeta \not{\partial} + m)\psi(x) + \frac{\lambda}{N} (\bar{\psi}(x)\psi(x))^2,$$

where ψ is a fermion field with N components (colors) and $m > 0$ is the fermion mass. In the original model proposed by Gross and Neveu in 1974 [96], the bare mass was 0, but the field “acquired a mass” by a complicated mechanism called *dynamical symmetry breaking*. This heuristic picture has been partially supported by results by Kopper, Magnen and Rivasseau for sufficiently large N [130], but we shall restrict the discussion to the massive case, for which definitive results have been attained.

In two spacetime dimensions the model is renormalizable and asymptotically free when $N > 1$. The (somewhat simplified) spatially and ultraviolet cutoff action is given by

$$\int_{\Lambda} \left[\frac{\lambda}{N} \left(\sum_{a=1}^N \bar{\psi}_a(x)\psi_a(x) \right)^2 + \delta m \left(\sum_{a=1}^N \bar{\psi}_a(x)\psi_a(x) \right) + \delta \zeta \left(\sum_{a=1}^N \bar{\psi}_a(x) i \not{\partial} \psi_a(x) \right) \right] d^2x$$

where δm is the mass counterterm, $\delta \zeta$ the wave function counterterm, and $N > 1$. The spinor indices and the ultraviolet cutoff subscripts are suppressed for transparency.

Taking advantage of the fact that perturbation theory for fermions is much simpler than that for bosons, Gawedzki and Kupiainen [75], on one hand, and Feldman *et alia* [65], on the other, provided different proofs that the Schwinger functions converge, as the volume and ultraviolet cutoffs are removed, to Schwinger functions satisfying the Osterwalder–Schrader axioms for all sufficiently small values of the renormalized coupling constant. Moreover, the limit Schwinger functions are the Borel sum of their standard renormalized perturbation series [65], *i.e.* the expansion in the renormalized coupling constant. By employing more recent advances in the rigorous treatment of renormalization group methods, Disertori and Rivasseau [48] have found a significantly simpler proof of these results and have shown that the mentioned Borel summability is uniform in N . Moreover, they demonstrate that the renormalization group equations and the β function are well

defined in the fully interacting limit theory. Iagolnitzer and Magnen [110] studied the Bethe–Salpeter kernel in this model and proved two-particle asymptotic completeness in this model.

The Gross–Neveu model has also been studied in 3 spacetime dimensions, where it is neither renormalizable nor asymptotically free. Heuristically, however, it is asymptotically free in the limit $N \rightarrow \infty$. After some changes of variables to represent the model for large N as a perturbation of its $N \rightarrow \infty$ limit and a rigorous summation of the simplest, most divergent coupling constant contributions that changes the nonrenormalizable model into a renormalizable one, de Calan, Faria de Veiga, Magnen and Sénéor have shown that for sufficiently large N the Schwinger functions of GN_3 exist after the ultraviolet and volume cutoffs are removed, though most of the Osterwalder–Schrader axioms have not yet been verified (*cf.* [39]). It is striking that the exact Schwinger functions exist but the standard perturbation series in this model does not, since it is not renormalizable.

3.6 Sine–Gordon₂ and Thirring models

In this section we discuss a set of models related to the sine–Gordon model, which is a model of a real, scalar massive or massless bosonic field ϕ in two spacetime dimensions with (spatially cutoff) action

$$V_\Lambda \equiv \zeta \int_\Lambda : \cos(\alpha\phi + \theta) : (x) d^2x,$$

where $\alpha, \zeta \in \mathbb{R}$ and $\theta \in [0, 2\pi)$. A number of different parametrizations of this action are to be found in the literature.

The ultraviolet divergences of this model depend on the size of $|\alpha|$. If $\alpha \in (-2\sqrt{\pi}, 2\sqrt{\pi})$, the model is superrenormalizable and already the measure

$$\mu_\Lambda \equiv \frac{e^{V_\Lambda} \mu_C}{\int e^{V_\Lambda(\phi)} d\mu_C(\phi)}$$

is well defined. In this range for α the model is amenable to the techniques previously developed to construct $P(\phi)_2$ models and models with exponential interaction in two spacetime dimensions. Fröhlich and Seiler [73] proved that for sufficiently small $|\zeta|/m_0^2$ the standard perturbation series expansion in ζ for the Schwinger functions actually converges in the infinite volume limit, the Schwinger functions satisfy the Osterwalder–Schrader axioms and the mass hyperboloid is isolated. The Haag–Ruelle scattering theory is applicable, and the S–matrix is nontrivial. Moreover, the perturbation series in ζ is asymptotic to the scattering amplitudes.

On the other hand, Park [156] showed that for the massless sine–Gordon model the infinite volume limits of the expectations (with respect to the corresponding finite volume measures) of products of (smeared) fields of the form $: \cos \epsilon(\phi + \theta) :$

$(x), : \sin \epsilon(\phi + \theta) : (x)$ and $\mathbf{a} \cdot \nabla \phi(x)$ (\mathbf{a} a constant vector) satisfy the Osterwalder–Schrader axioms (with the possible exception of clustering) for all parameter values in the indicated ranges by using suitable correlation inequalities.

When $\alpha^2 \in [4\pi, 8\pi)$ the model is superrenormalizable, but the number of renormalization counterterms increases to infinity as α increases to $8\pi^2$. Nicolò, Renn and Steinmann [148] proved that in a finite volume the massive sine–Gordon model is ultraviolet stable for α in this range. In the same range Dimock and Hurd use renormalization group methods to prove bounds on the finite volume Schwinger functions for both the massless and massive cases which are uniform in Λ in the massive model. Moreover, they show that in a finite volume the Schwinger functions are analytic in ζ at 0. However, the full control of the infinite volume limit has not yet been attained.

When $\alpha^2 = 8\pi$ the model is heuristically renormalizable but not superrenormalizable, and when $\alpha^2 > 8\pi$ it is nonrenormalizable. Nicolò and Perfetti [149] have proven that for $\alpha^2 = 8\pi$ the model is indeed perturbatively renormalizable. And for $\alpha^2 > 8\pi$ Dimock and Hurd [47] have shown that the model is asymptotically free in the infrared. However other rigorous results in these cases are not yet available.

One of the first manifestations of the equivalence between apparently distinct quantum field models is the phenomenon of *bosonization* in two spacetime dimensions. A simple example is the equivalence between free massless fermionic fields and free massless bosonic fields given by the identifications

$$\bar{\psi}(1 + \sigma\gamma_5)\psi \leftrightarrow c : e^{i\sigma\sqrt{4\pi}\phi} : , \quad \bar{\psi}\gamma^\mu\psi \leftrightarrow -\frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu\phi ,$$

where $\sigma = \pm 1$, ϵ is the standard totally antisymmetric matrix with entries $\pm 1, 0$, and c is a suitable constant depending on the choice of Wick product used. Coleman [42] gave a heuristic argument suggesting a similar equivalence between the massive Thirring model with Lagrangian

$$Z\bar{\psi}i\not{\partial}\psi - \tau Z_1\bar{\psi}\psi - \frac{\lambda}{4}Z^2j_\mu j^\mu ,$$

where Z, Z_1 are renormalization constants and $j_\mu = \bar{\psi}\gamma^\mu\psi$, and the massless Sine–Gordon model with Lagrangian

$$\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \zeta : \cos(\alpha\phi) :$$

under the identifications

$$Z_1\bar{\psi}(1 + \sigma\gamma_5)\psi \leftrightarrow c : e^{i\sigma\alpha\phi} : , \quad Z\bar{\psi}\gamma^\mu\psi \leftrightarrow -c_1\epsilon^{\mu\nu}\partial_\nu\phi ,$$

where c, c_1 are suitable constants depending on λ and the ultraviolet cutoff used. In order for this equivalence to be valid, certain relations among the Thirring parameters λ, τ and the Sine–Gordon parameters ζ, α must obtain. The case

$\alpha^2 = 4\pi$ corresponds to free fermions ($\lambda = 0$), and the case $\zeta = 0$ (free bosons) corresponds to massless fermions ($\tau = 0$).

Closely associated with these models is the Thirring–Schwinger model, often referred to as QED₂ since the only manifestation of electromagnetism in one spatial dimension is the Coulomb force between charges. The interaction Lagrangian for this model is

$$\frac{g}{2}j_\mu j^\mu + \frac{\pi e^2}{2}\tilde{j}^0 V \tilde{j}^0,$$

where ψ is a massive two-component Fermi field, j_μ is the conserved current defined above, and $\tilde{j}^0 = j^0 + j_c^0$, with j_c^0 is a formal c -number current specifying charges at infinity, and $V = \frac{1}{2}|x|$ is the one-dimensional Coulomb potential. Fröhlich and Seiler [73] rigorously proved the equivalence between this model and the massive sine-Gordon model for sufficiently large mass and $\alpha^2 < 4\pi$ by showing, along the lines of Coleman’s original argument, that their perturbation series coincide term by term and actually converge.

From the point of view of heuristic Lagrangian considerations, the Thirring model is renormalizable but not superrenormalizable. In point of fact, different versions of the (massless) “Thirring model” are extant. There is the original model introduced by Thirring [191], for which Glaser [80] found an explicit “solution” for the fields. However, Ruijsenaars [161] showed that the n -point functions of the fields of this “solution” do not exist. Another concrete realization was proposed by Johnson [118], which, although it is mathematically unsatisfactory, led to a rigorous formula by Klaiber [124] for Wightman functions which coincide with the n -point functions of Johnson’s realization for $n = 2, 4$. Carey, Ruijsenaars and Wright [41] proved that Klaiber’s functions satisfy the Wightman axioms. In fact, they rigorously constructed the fields corresponding to these n -point functions as strong limits of certain natural approximating fields.

A rigorous construction of the massive Thirring model was finally achieved by Benfatto, Falco and Mastropietro [19]. Writing the ultraviolet cutoff generating functional for the Euclidean model as a Grassmanian integral, they showed that after proper choice of the wave function renormalization and bare mass the Schwinger functions (at noncoinciding points) converge as the cutoff is removed to Schwinger functions satisfying the Osterwalder–Schrader axioms. A multiscale renormalization group approach related to the one discussed in Section 3.3 is used in the proof, and so the results are valid for sufficiently small values of the coupling constant (there is no restriction on the mass). Curiously, they also find that in the massless case the resultant two-point function differs from that found by Johnson. Benfatto, Falco and Mastropietro [20] then went on to prove Coleman’s equivalence between the massless sine–Gordon model and the massive Thirring model in any finite volume.

3.7 Local gauge quantum field theories

Local gauge quantum field theories²² are conceptually and mathematically quite challenging, and we discuss briefly some of these challenges in this section. Although we are concerned here with CQFT, it is necessary to treat some of the insights gained into the nature of gauge theories by other mathematically rigorous means, since they shed light on the nature of these challenges and on the consequences for the mathematical framework of the models. But even the mathematically rigorous literature on aspects of local gauge theory has become enormous, so only a few highlights can be discussed here.

One of the challenges facing the rigorous construction of such models is to decide what constitutes the proper framework for the end result. It has become clear that, though the Wightman axioms are still suitable for the *gauge invariant local* fields in gauge theories, such as the electromagnetic fields $F^{\mu\nu}$, they must be supplemented to include extended field objects, since it is also useful to consider gauge invariant objects such as

$$\overline{\psi}(x) e^{i \int_{C_{xy}} A_\mu(z) dz^\mu} \psi(y), \quad (3.8)$$

here C_{xy} is a suitable curve connecting the points x and y , A is a gauge potential, and ψ a fermionic field. Various incomplete proposals have been made in this regard, but the most developed of these appears to be that of Fröhlich, Osterwalder and Seiler (*cf.* the last chapter of [172]), which provides conditions on expectations of products of Euclidean Wilson loops $e^{i \int_C A_\mu(x) dx^\mu}$ for piecewise smooth loops C in spacelike hyperplanes in Minkowski space (viewed, however, in Euclidean space) as well as a procedure to construct the corresponding Minkowski space theory. Another proposal by Ashtekar, Thiemann and co-workers [11, 190] places its conditions instead on the Euclidean measure of the model, instead of on specific classes of expectations. These conditions also afford the possibility to reconstruct important aspects of the real time theory from the Euclidean data. They have shown that a large class of Yang–Mills models in two spacetime dimensions verify these conditions — see Section 3.7.3.

In addition, it is convenient for various purposes to include the unobservable gauge potentials A_μ in heuristic gauge QFT. But if they are allowed to enter into the theory as a quantum field in their own right, the price is high. It was recognized by many physicists that the introduction of gauge potentials as quantum fields acting on the state space is incompatible with both Lorentz covariance and Einstein causality. We illustrate this point with the electromagnetic field in Section 3.7.2. The upshot is that either one must choose a gauge in which the gauge potential acts as an operator in a Hilbert space but is not Lorentz covariant and violates

²²The adjective “local” modifies “gauge”, not “quantum field theory”. The internal symmetry groups in these theories are infinite dimensional, as the “gauge transformation” can depend upon the spacetime point at which it is being implemented. This is to be contrasted with “global gauge groups”, such as the global $U(1)$ symmetry in the Yukawa models or the global \mathbb{Z}_2 symmetry in $P(\phi)_2$ models with even polynomial P .

Einstein causality, or one must choose a gauge in which the gauge potential is Lorentz covariant and satisfies Einstein causality but acts as an operator in a vector space with an indefinite inner product, *i.e.* there exist vectors Ψ in the state space such that $\langle \Psi, \Psi \rangle < 0$. In such a circumstance the standard relation between quantum expectations and probabilities fails. One should note that perturbation theory is generally performed in gauges of the latter type. Additional complications are introduced by the (again unobservable) charge carrying Fermi fields in such quantum field theories (again illustrated in Section 3.7.2).

A modified version of the Wightman axioms which takes into account the necessity of employing a reasonably behaved indefinite inner product space called a Krein space may be found in the monograph of Strocchi [180].²³ The essential modification is that the positivity condition on the Wightman functions (or the reflection positivity condition on the Schwinger functions) is replaced by a “Hilbert space structure” condition which permits the construction of subspaces of the Krein space suitably associated with the Wightman functions on which the inner product is positive definite.

Taking the standpoint that the Euclidean gauge potentials are to be taken explicitly into account as dynamical variables brings up the question of which measure to adopt on the space of potentials. In the geometric formulation of classical electrodynamics, the gauge potentials are connections on a certain fiber bundle with base space \mathbb{R}^d (and the Wilson loops are (traces of) holonomies of the connection around closed loops). Let \mathcal{A} denote the set of such connections, and let \mathcal{G} denote the local gauge group acting on \mathcal{A} . In gauge theories it is natural to view each gauge equivalence class of connections as a distinct physical “path”, *i.e.* the orbit of any single potential $A \in \mathcal{A}$ under the action of \mathcal{G} is an equivalence class. The elements of each orbit are thus viewed as physically equivalent. However, the resultant path space — the space of all such orbits — is a nonlinear quotient space \mathcal{A}/\mathcal{G} ; this introduces technical difficulties. By using a gauge fixing one can impose a linear structure on \mathcal{A}/\mathcal{G} for $d = 2$, but in higher dimensions, the resultant Gribov ambiguities limit the usefulness of such gauge fixings.

A further complication arising when one takes the gauge potentials as dynamical variables is that the measures of physical relevance in QFT are typically not supported on functions having nice properties from the point of view of analysis (in particular, the measures are not supported on \mathcal{A}/\mathcal{G}), so that one must introduce a suitable closure of \mathcal{A}/\mathcal{G} in which one can find the relevant generalized connections on which these measures do have support. The choice of this closure is not unique, and many can be found in the literature. The measures are then defined on this closure. In the scalar and fermionic Euclidean models described above, the (cutoff) physical measure is defined as a suitable perturbation of a Gaussian measure corresponding to a free field. In CQFT this Gaussian measure replaces

²³A modified version of the Osterwalder–Schrader axioms for the same purpose has been proposed by Jakobczyk and Strocchi [114]. They show that if the Schwinger functions satisfy these modified conditions, then the reconstructed Wightman functions satisfy the modified Wightman axioms.

the “measure” $\exp\{-m_0^2 \int \phi^2(x) dx^d\} d\nu(\phi)/Z$ to be found in heuristic treatments of functional integration in QFT, where ν is the nonexistent Lebesgue measure on $\mathcal{S}'(\mathbb{R}^d)$. For gauge models Ashtekar and Lewandowski [10] have constructed a uniform Borel measure μ_0 on a natural closure of \mathcal{A}/\mathcal{G} which is gauge invariant and strictly positive on continuous cylindrical functions. This measure replaces ν in many rigorous treatments of functional integration for gauge theories. The physical measures are then obtained by suitably perturbing μ_0 . This is illustrated in Section 3.7.3. The advantage of this approach is that the gauge invariance is explicit at every step. Any Fermi fields coupled to the gauge potentials would be “integrated out” along the lines of the previous work on the Yukawa models, resulting in a new effective action perturbing the pure gauge measure.

The complications in defining measures on such path spaces are great, leading many researchers to concentrate on the topological and geometric aspects of the problems (in so-called topological quantum field theory), largely bypassing (or ignoring) the analytical aspects. These interesting developments are not treated here. And, although also classical gravitation can be understood as a local gauge theory, we do not address any aspect of quantum gravitation here. However, we give a brief accounting of the constructive results concerning the physically most important quantum gauge models, even though definitive results have been attained only in spacetime dimensions less than 4 and, as indicated above, aspects of these models do not conform to the Wightman setting. It should be emphasized that it is not at all clear that gauge theories such as QED and QCD cannot be incorporated into the HAK setting, since there the observable quantities are primary and many of the problems we have discussed above arise only after introduction of unobservable fields. One of the many successes of AQFT is that Doplicher, Haag and Roberts have shown how, starting with the (necessarily gauge invariant) observables in a *global* gauge theory, one can uniquely *derive* both the gauge group and the associated charge carrying fields — *cf. e.g.* [9, 104]. Although a corresponding breakthrough for *local* gauge theories has not yet been achieved, the possibility remains open.

3.7.1 Abelian Higgs_d model

The abelian (or $U(1)$) Higgs model is an interacting theory of a vector field $A_\nu(x)$ coupled in a gauge covariant manner to an N -component scalar field $\phi(x)$ and is sometimes referred to as scalar electrodynamics (when $N = 1$). The Euclidean action of the model in d spacetime dimensions is

$$\int \left(\frac{1}{4} \sum_{\mu, \nu=1}^d |F_{\mu\nu}(x)|^2 + \frac{1}{2} \sum_{\mu=1}^d |D_\mu \phi(x)|^2 + \frac{1}{2} m^2 |\phi(x)|^2 + \lambda |\phi(x)|^4 \right) d^d x,$$

where the field strength tensor is $F_{\mu\nu}(x) = \partial_\mu A_\nu - \partial_\nu A_\mu$, and the covariant derivative of the scalar field is

$$(D_\mu \phi)_i(x) = \partial_\mu \phi_i(x) - e A_\mu(x) (Q\phi)_i(x).$$

Q is an antisymmetric $N \times N$ matrix, and e and λ are coupling constants. For $d = 2, 3$ the model is superrenormalizable.

In a series of papers (see [31] for references), Brydges, Fröhlich and Seiler studied this model in two spacetime dimensions by commencing with the model on a Euclidean lattice. For the convenience of avoiding spurious infrared divergences, the bare mass of the gauge field A_μ is initially taken to be strictly positive, but ultimately the continuum limit, the ultraviolet limit and the limit in which the bare mass of the gauge field tends to zero are all controlled, and all the Osterwalder–Schrader axioms except clustering are verified for Schwinger functions involving *gauge invariant* local fields such as $|\phi|^2$ and $F_{\mu\nu}$, as well as string and loop observables such as

$$: \bar{\phi}(x) e^{\int_x^y A_\mu dx'_\mu} \phi(y) :$$

and $: e^{\oint A_\mu dx_\mu} :.$ Hence, there exists a corresponding Wightman theory for these fields, though the vacuum may not be unique.

King [122, 123] studied the abelian Higgs model in two and three spacetime dimensions. He also began with the model on a finite volume lattice and controlled the continuum and infinite volume limits using a combination of renormalization group techniques and correlation inequalities. He verified all of Osterwalder–Schrader axioms except ergodicity and the regularity properties which assure that the Wightman functions are tempered distributions.

Balaban, Imbrie and Jaffe [15] have examined the abelian Higgs model for $d = 2, 3$ for rigorous evidence supporting the heuristic notion of the Higgs mechanism, *i.e.* the notion that, under certain circumstances usually associated with spontaneous breaking of the gauge symmetry, a massless particle can acquire mass. A proof of the existence of a mass gap on the Hilbert space generated from the vacuum by application of all gauge invariant observables could be taken as such supporting evidence. Together with Brydges they supplied such a proof for the model on a lattice. Although they made much progress on the proof of the mass gap in the continuum limit [15], the final step in that proof never appeared in print.

3.7.2 Quantum electrodynamics_d

Despite the success of the experimental predictions made by the perturbation theory computations associated with QED, it is widely believed that since QED is not asymptotically free in four spacetime dimensions it cannot be defined as a mathematically rigorous theory; but this matter has not been settled. Moreover, beyond the ultraviolet problems inherent in the model some serious conceptual difficulties remain to be resolved. As a first example, though the notion of a gauge transformation is straightforward in classical electrodynamics, it is not at all clear what gauge transformations are in QED²⁴ and what their relation may

²⁴as evidenced, for instance, by an investigation by Strocchi and Wightman [178]

be to the classical gauge transformations which play such an important role in the semi-classical reasoning commonly found in heuristic QFT.

In addition, as shown by a number of mathematical physicists (see Strocchi's monograph [180] for details and references), under various sets of reasonable assumptions, the standard picture in quantum theory of a Hilbert space \mathcal{H} serving as the state space for the model is inconsistent with each of the following: the potential field A_μ is covariant, A_μ satisfies Einstein causality, and Maxwell's equations are satisfied on \mathcal{H} . Wightman and Gårding [196] have shown that for the free electromagnetic field a formalism due to Gupta and Bleuler is mathematically consistent. In particular, there exists a vector space \mathcal{H} with a sesquilinear Hermitian form $\langle \cdot, \cdot \rangle$ on which acts a unitary representation $U(\mathcal{P}_+^\uparrow)$ of the Poincaré group satisfying the spectrum condition and operator valued tempered distributions $A_\mu(x)$ and $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$, which are covariant under the action of $U(\mathcal{P}_+^\uparrow)$ and satisfy Einstein causality. In \mathcal{H} is a distinguished subspace \mathcal{H}' such that $\langle \Psi, \Psi \rangle \geq 0$ for all $\Psi \in \mathcal{H}'$ and in which Maxwell's equations hold:

$$\langle \Phi, \partial^\mu F_{\mu\nu}(x) \Psi \rangle = 0$$

for all $\Phi, \Psi \in \mathcal{H}'$. The (pure) physical states are described by unit vectors in the Hilbert space given by the quotient $\mathcal{H}'/\mathcal{H}''$, where \mathcal{H}'' is the subspace of \mathcal{H}' consisting of vectors Ψ such that $\langle \Psi, \Psi \rangle = 0$. For these the standard quantum probability interpretation is valid.

Though there are a number of realizations of the Euclidean free electromagnetic field in spacetime dimension d , the most concise is to specify its generating function:

$$f \mapsto \exp \left[-\frac{g^2}{2} \langle (-\Delta_d)^{-1} \delta f, \delta f \rangle \right], \quad (3.9)$$

where Δ_d is the Laplacian on \mathbb{R}^d , $(\delta f)_\nu(x) = \sum_\mu \partial f_{\mu\nu}(x) / \partial x^\mu$ and f is a tempered function valued 2-form. This satisfies the hypotheses of Minlos' theorem, so there exists a probability measure μ on the dual space of the space of test 2-forms such that $\int \exp[iF(f)] d\mu$ equals (3.9). The Euclidean free electromagnetic field can be defined as the generalized stochastic process $f \mapsto F(f)$ satisfying that equation. The corresponding Schwinger functions satisfy all of the Osterwalder-Schrader axioms (note that the gauge potentials do not appear at all). The exterior derivative of F is zero. Realizations of the Euclidean free electromagnetic field involving the gauge potentials run into the problems indicated above.

Additional complications are introduced when one considers the electron field $\psi(x)$. Because the photon is massless and Gauss' Law holds, the physical electron field carrying the electric charge cannot be Lorentz covariant or satisfy Einstein causality. Moreover, in a Hilbert space \mathcal{H} the electron field $\psi(x)$ cannot satisfy Einstein causality if Maxwell's equations with a source hold in \mathcal{H} . Furthermore, in fully interacting QED the physical state space will be complicated not only by the considerations mentioned above and by the superselection rule associated with the global electric charge operator, resulting in countably infinitely many distinct charge sectors, but also by the uncountably infinitely many sectors associated with

distinct asymptotic configurations of soft photons. (For a profound investigation of these matters, the reader is referred to a paper of Buchholz [33].) In heuristic QED this abundance of superselection sectors is ignored, and one selects more or less arbitrarily a subset of physical states to work with, typically discarding in this manner the charged states with the best possible localization properties.

The perturbation theory for QED is on solid mathematical footing. QED is renormalizable in four spacetime dimensions (see, in particular, the proof in [66]), so the perturbation series is well defined to all orders. Indeed, it has been proven by Feldman and co-authors [66] that the standard perturbation series for the Schwinger functions in QED₄ is locally Borel summable. It is not known if there exists an exact model to whose Schwinger functions this series is asymptotic. So we turn now to lower dimensions.

Weingarten and Challifour [193] formulated QED₂ on a finite lattice and showed that a Salam–Matthews formula for Schwinger functions analogous to (3.7) holds, where the reference measure is that of the lattice gauge theory discussed in Section 3.7.3 with gauge group $G = \text{U}(1)$ (which gives a lattice approximation to the action of free Euclidean electromagnetic fields expressed in terms of the electromagnetic potentials), the product of scalar fields is replaced by a polynomial of electromagnetic potentials and the renormalized determinant is replaced by a similar expression. They showed [193, 194] that in the limit as the lattice spacing goes to 0 and the volume cutoff is removed, their expressions for the Schwinger functions involving arbitrary products of the gauge potentials and products of pairs of fermionic and antifermionic fields have well defined limits. However, none of the Osterwalder–Schrader axioms were addressed. Seiler [172] has given a sketch of a cluster expansion whose convergence would verify all the Osterwalder–Schrader axioms for the Schwinger functions involving gauge invariant local fields (such as the electromagnetic field and the fermion current) plus a mass gap for the fermion (electron) for sufficiently small ratio of the electric charge to the fermion mass, but the details have not appeared in print.

3.7.3 Yang–Mills_d models

There are many more or less equivalent formulations of the classical Yang–Mills theory. The most concise, which requires familiarity with some basic concepts of differential geometry, is the following: Let G be a compact group, A be a connection on a G -bundle over \mathbb{R}^d (A is a 1-form taking values in the Lie algebra \mathfrak{G} of G), and $F = dA + A \wedge A$ be the corresponding curvature. The (classical) action for the pure Yang–Mills_d model is

$$\frac{1}{4g^2} \int_{\mathbb{R}^d} \text{Tr} F \wedge *F,$$

where Tr denotes an invariant quadratic form on \mathfrak{G} and g plays the role of a coupling constant. Introducing coordinates on \mathbb{R}^d and a basis in \mathfrak{G} , one has

$$F_{\mu\nu,a} = \partial_\mu A_{\nu,a} - \partial_\nu A_{\mu,a} - c_a^{bc} A_{\mu,b} A_{\nu,c},$$

where c_a^{bc} are the structure constants of \mathfrak{G} . In these terms the action is

$$-\frac{1}{4g^2} \int_{\mathbb{R}^d} \sum_a F_{\mu\nu,a}(x) F_a^{\mu\nu}(x) dx^d.$$

If $G = U(1)$, then the structure constants are 0 and one recovers the action of pure electromagnetism. In Yang–Mills theories G is understood to be nonabelian. For example, in QCD one takes $G = SU(3)$ and in the electroweak theory of Glashow, Salam and Weinberg one takes $G = SU(2) \times U(1)$ (both of these are components of the SM). In QCD and the SM there are, in addition, matter fields which are coupled to the gauge potentials A and provide a further contribution to the action. These are not discussed here, with one exception, since the rigorous results in these cases are minimal at this point in time. However, there are results of interest concerning pure Yang–Mills models, and we turn to those next.

For spacetime dimension $d > 2$ one is faced again with ultraviolet problems that go beyond Wick ordering. And since almost all means of regularization (*i.e.* suppressing high energy–momentum values) destroy gauge invariance, most rigorous studies of (Euclidean) Yang–Mills in higher dimensions commence with the theory on a lattice. There are various versions of this, but we will consider only one, due essentially to Wilson [197].

Let Λ be a hypercubical lattice in \mathbb{R}^d with lattice spacing a . Ordered pairs xy of nearest-neighbor lattice points $x, y \in \Lambda$ are called bonds, closed loops $uvxy$ consisting of the obvious four bonds are called plaquettes. Let G be a compact Lie group, let χ be a character on G , and let g_\cdot be a map from the bonds in Λ into G such that $g_{xy} = g_{yx}^{-1}$. The collection of such maps is called the field configuration space. To each plaquette $P = uvxy$ in Λ corresponds the conjugacy class g_P of the element $g_{uv}g_{vx}g_{xy}g_{yu}$ and the quantity $A_P = \frac{1}{2}(\chi(g_P) + \overline{\chi}(g_P))$. The Wilson action²⁵ on the lattice Λ is defined in terms of these quantities:

$$A_\Lambda^W \equiv \frac{1}{g^2} \sum_\Lambda A_P,$$

where the sum runs over all plaquettes P in Λ . A probability measure on the field configuration space is given by

$$\mu_\Lambda \equiv \frac{e^{-A_\Lambda^W} \prod dg_{xy}}{\int e^{-A_\Lambda^W} \prod dg_{xy}},$$

where the product is taken over all bonds xy in the lattice and dg_{xy} is the Haar measure on G for each such bond. Gauge transformations are given by $g_{xy} \mapsto \gamma_x g_{xy} \gamma_y^{-1}$, where $\gamma : \Lambda \rightarrow G$. Both the action and the measure are gauge invariant. As shown by Osterwalder and Seiler [154], the expectations with respect to μ_Λ of gauge invariant functions on the field configuration space satisfy a natural analogue

²⁵Another, less commonly used action for lattice gauge theories is the Villain action, which is an approximation to the Wilson action. Its description is lengthier and is not presented here.

of the reflection positivity condition in the Osterwalder–Schrader axioms, so that one can construct a corresponding Hilbert space and a positive self-adjoint transfer matrix (also bosons and fermions can be naturally incorporated into this setting [154]). Much rigorous work has treated the infinite volume limit of such theories, but for the purposes of quantum field theory one also must control the limit as $a \rightarrow 0$. For this, there are significantly fewer results.

Gross [97] showed that as the lattice spacing converges to 0 the $U(1)_3$ lattice gauge model with the Villain action converges to the free Euclidean electromagnetic field in the sense of convergence of the characteristic functions of the field variables $F_{\mu\nu}$. He also showed that if the Wilson action is adopted, then the characteristic function of the “lattice current” converges to the characteristic function of the current of the free Euclidean electromagnetic field as the lattice spacing goes to 0. Driver [53] proved that similar results hold for the $U(1)_4$ lattice gauge model.

In two spacetime dimensions, when the complete axial gauge is taken the Euclidean Yang–Mills action simplifies sufficiently to define a Gaussian measure for the gauge field. This greatly reduces the difficulties in the task of construction and affords the possibility of avoiding lattice approximations (though at the cost of giving up explicit gauge invariance), and a number of different approaches have been developed to that end. Only a few highlights can be mentioned here. Gross, King and Sengupta [98] constructed the Euclidean $U(N)$ pure Yang–Mills model in the axial gauge in two spacetime dimensions using stochastic differential equations (*cf.* Section 5). They show that the differential equation controlling parallel transport along a smooth curve in \mathbb{R}^2 with respect to a typical gauge potential can be interpreted as a stochastic differential equation. Without cutoffs they construct the Schwinger functions for the (nonoverlapping) Wilson loops, provide closed expressions for these Schwinger functions and verify their Euclidean invariance. Driver [54] extended their results to include expectations of more general functions of parallel transport along a finite “admissible collection” of curves and further proved that the continuum model is the limit of the (gauge fixed) lattice approximations (for both the Villain and the Wilson actions) as the lattice spacing goes to 0. Anshelevich and Sengupta [6] have shown that the limit $N \rightarrow \infty$ (with N/g^2 held fixed) of these Euclidean $U(N)$ pure YM_2 models exists and is described by a concrete free stochastic process.

These results have been extended in many respects by Ashtekar and co-authors [11]. Studying Euclidean pure YM_2 models with G equal to $SU(N)$ or $U(1)$, they take the reference measure μ_0 on a suitable closure of \mathcal{A}/\mathcal{G} discussed above, perturb it by $\exp\{-A_\Lambda^W\}$ (see below), normalize to obtain a probability measure, and explicitly compute the expectations of products of Wilson loops. They then show that the resulting expressions have a well defined limit as the volume cutoff is removed and the lattice spacing goes to 0.²⁶ They therefore have rigorously established a closed expression for the expectations of products of generic Wilson

²⁶Fleischhack [68] has repaired some technical problems in their work, recovering their results.

loops for YM_2 . In addition, they explicitly construct the corresponding real time model and establish the equivalence between the Euclidean and real time formulations. Klimek and Kondracki [128] have constructed a measure without cutoffs for Euclidean YM_2 with $G = \text{SU}(2)$ coupled to a massive fermion, but they did not verify any of the usual axioms.

There has been progress in the rigorous analysis of YM_4 , primarily for $G = \text{SU}(N)$, though in the few places where the details of the group make a difference in the proof of the estimates $G = \text{SU}(2)$ is usually taken. In a highly technical argument incorporating renormalization group transformations and stretching over several lengthy papers, Balaban (see [16] for references) succeeded in proving the ultraviolet stability of the model on a lattice of arbitrary bond length. This should be the core of a proof that the ultraviolet limit (here in the guise of the lattice spacing going to 0) of the expectations of products of Wilson loops exists. However, the series of papers came to an end before exact Schwinger functions were constructed and their properties were verified. Unfortunately, the same is true of a series of papers by Federbush on the same model (see [62] for references).

The problem of constructing a pure Yang–Mills model in four spacetime dimensions remains open. In fact, the Clay Mathematics Institute has offered one million dollars to anyone who succeeds in showing that for any compact simple group G the corresponding pure Yang–Mills model in four spacetime dimensions exists, satisfies conditions at least as strong as the Wightman or Osterwalder–Schrader axioms, is nontrivial and manifests a mass gap.

4 Functional Integral Constructions — Minkowski

Real time functional integrals are technically more difficult to work with than the corresponding Euclidean integrals, since they are oscillatory in nature and do not benefit from exponential suppression of large ranges of field “values”, as do the Euclidean integrals. Although much progress has been made in the control of the functional integrals arising in nonrelativistic quantum mechanics, the situation for the real time functional integrals that are relevant to QFT is much less developed. The reader is referred to the monograph of Klauder [125] for an accounting of both. There is not yet sufficient progress to treat interacting relativistic quantum fields rigorously in this framework.

5 Constructions Using Stochastic Differential Equations

The strong links between Euclidean QFT and probability theory have been suggested in Section 3, but we discuss another such link in this section. If B_t is a cylindric version of $\mathcal{S}'(\mathbb{R}^{d-1})$ -valued Brownian motion, so that for each $f \in \mathcal{S}(\mathbb{R}^{d-1})$

$b_t^f \equiv B_t(f)$ is a version of one dimensional Brownian motion, then the linear Ito stochastic differential equation (SDE)

$$d\xi_t^0 = \sqrt{-\Delta_{d-1} + 1} \xi_t^0 dt + dB_t \quad (5.1)$$

has a stationary Gaussian solution ξ_t^0 with mean 0 and expectation which coincides with that of the free Euclidean scalar Bose field with mass 1:

$$E\xi_0^0(\mathbf{x})\xi_t^0(\mathbf{y}) = \int_{\mathcal{S}'(\mathbb{R}^d)} \phi(0, \mathbf{x})\phi(t, \mathbf{y}) d\mu_{C_1}.$$

These stochastic processes can therefore be identified in the indicated sense. Another such relation of note is provided by considering a Gaussian white noise η on $\mathcal{S}'(\mathbb{R}^d)$, *i.e.* η is a $\mathcal{S}'(\mathbb{R}^d)$ -valued random variable distributed according to a Gaussian measure ν with generating function

$$\int_{\mathcal{S}'(\mathbb{R}^d)} e^{i\eta(f)} d\nu(\eta) = \exp(-\frac{1}{2}\|f\|_2^2).$$

For $\lambda \in (0, \frac{1}{2}]$, the solution of the stochastic partial (pseudo-)differential equation (SPDE)

$$(-\Delta + 1)^\lambda \phi_\lambda = \eta$$

is given by the stochastic convolution integral

$$\phi_\lambda = (-\Delta + 1)^{-\lambda} * \eta.$$

ϕ_λ is a Gaussian stochastic process of mean 0 and covariance

$$E\phi_\lambda(x)\phi_\lambda(y) = (-\Delta + 1)^{-2\lambda}(x - y),$$

so that $\phi_{1/2}$ can be identified with the free Euclidean scalar Bose field with mass 1.

To get beyond Gaussian processes (and thus free quantum fields), essentially two approaches have been introduced. One is to perturb the linear drift term in (5.1) with a nonlinear term. In this case the typical sample paths are distributions instead of functions. The second is to change the reference Gaussian process (Brownian motion or white noise) into a suitable non-Gaussian process, for example to perturb the white noise with Poisson noise. In both one typically finds a stochastic process which is a weak solution of the SDE one has set up; then the equilibrium measure corresponding to this process is the measure sought in the Euclidean QFT program.

These methods or their variants have been employed by many authors for the purpose of constructing quantum field models. As an example, we summarize work of Jona-Lasinio and Mitter [119]. Let Λ be a square region in \mathbb{R}^2 , Δ_Λ be the Laplace operator with Dirichlet boundary conditions on Λ and $C = (-\Delta_\Lambda + 1)^{-1}$. For $\varepsilon \in (0, \frac{1}{10})$ consider the nonlinear SDE

$$d\hat{\phi}_t = -\frac{1}{2}(C^{-\varepsilon}\hat{\phi}_t + \lambda C^{1-\varepsilon} : \hat{\phi}_t^3 :) dt + dW_t \quad (5.2)$$

with some specified initial condition ι ,²⁷ where W_t is a Brownian motion with mean 0 and covariance

$$E W_t(f) W_s(g) = \langle f, C^{1-\varepsilon} g \rangle \min\{t, s\}.$$

Jona-Lasinio and Mitter construct a Markov process $\hat{\phi}_t$ which is a weak solution of (5.2), is ergodic and mixing, and satisfies

$$\lim_{t \rightarrow \infty} E_{\hat{\phi}_0 = \iota} \hat{\phi}_t(f_1) \dots \hat{\phi}_t(f_n) = \int \phi(f_1) \dots \phi(f_n) d\mu_\Lambda(\phi),$$

where μ_Λ is the measure given in (3.1) with $P(\phi) = \phi^4$ and C the covariance mentioned directly above. Hence, they have constructed, in this sense, the ϕ_2^4 model in a finite volume. Similar constructions have been made of the $P(\phi)_2$ models, the sine-Gordon model (in the range $\alpha^2 < 4\pi$) and the exponential interaction model in two spacetime dimensions, all with a volume cutoff (see *e.g.* [198]).

The process $\hat{\phi}_t$ constructed in this manner is Λ -dependent. Of particular interest is the fact that Borkar, Chari and Mitter [25] showed that one can take the infinite volume limit also on the left hand side of the above equality (the processes indexed by Λ converge in a suitable sense as $\Lambda \nearrow \mathbb{R}^2$). Since it is already known that the limit of the right hand side as $\Lambda \nearrow \mathbb{R}^2$ yields the Schwinger functions of the fully interacting ϕ_2^4 model, one sees that this approach also succeeds in constructing the exact ϕ_2^4 model. The methods of Borkar, Chari and Mitter should also be applicable to general $P(\phi)_2$ models.

Some effort has been expended to understand renormalization from the point of view of stochastic quantization. However, this has not yet resulted in a construction of a quantum field model requiring an ultraviolet renormalization going beyond Wick ordering.

Albeverio and co-workers have constructed a class of Euclidean random fields via convolution from generalized white noise and have shown that the Schwinger functions of these models can be analytically continued to real time “Wightman functions” (*cf.* [4] for details and references). All of the Wightman axioms hold for these models except possibly the positivity condition²⁸ on the family of Wightman functions that allows one to construct a Hilbert space in which the field operators act. Albeverio, Gottschalk and Wu [4] have shown that their “Wightman functions” satisfy a weakened condition due to Morchio and Strocchi (*cf.* [180]) motivated by the study of local gauge theories (see Section 3.7). This condition assures that one can construct a Krein space, instead of a Hilbert space, in which the special subset \mathcal{D} mentioned in Section 1 is dense. In these models one may choose d as one likes, so this approach results in QFT models in four spacetime dimensions represented in Krein spaces, instead of Hilbert spaces. The scattering behavior in such models has also been studied by Albeverio and Gottschalk [5], and some of the models manifest nontrivial scattering. However, it is not clear at this point whether these models include any of the models of interest in elementary particle physics.

²⁷The solution and its laws will therefore depend on the choice of ι .

²⁸In some instances it has been proven that the positivity condition is, in fact, violated.

6 Algebraic Constructions II

Relatively recently, new insights and tools attained in AQFT have led to an important renaissance of algebraic real time constructions of quantum field models, resulting in new techniques and the construction of models which either cannot be constructed by other known techniques or can only be so constructed with a prohibitive amount of effort. In these constructions one is not guided by Lagrangian QFT but rather uses other input to arrive at the models.

For many of these constructions a particular collection of spacetime regions is distinguished — the so-called wedges. After choosing an inertial frame of reference in Minkowski space, one defines the right wedge to be the set $\mathcal{W}_R = \{x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4 \mid x_1 > |x_0|\}$ and the set of wedges to be $\mathbf{W} = \{\lambda \mathcal{W}_R \mid \lambda \in \mathcal{P}_+^\uparrow\}$. The set of wedges is independent of the choice of reference frame; only which wedge is designated the right wedge is frame-dependent. Let $\theta_{\mathcal{W}}$ denote the reflection on Minkowski space about the “edge” of the wedge \mathcal{W} . The set $\{\theta_{\mathcal{W}} \mid \mathcal{W} \in \mathbf{W}\}$ generates the proper Poincaré group \mathcal{P}_+ . These regions have been of interest since an important insight won by Bisognano and Wichmann [21], namely that for any net $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O} \in \mathcal{R}}$ of von Neumann algebras locally associated to a Wightman theory $(\phi, \mathcal{H}, U, \Omega)$ in a natural manner²⁹ the abstract modular objects $\Delta_{\mathcal{W}}$, $J_{\mathcal{W}}$ associated with the pair $(\mathcal{A}(\mathcal{W}), \Omega)$ by Tomita–Takesaki theory [26, 185, 189] have, in fact, physical meaning. They showed that one has

$$J_{\mathcal{W}_R} = \Theta U(R_1(\pi)), \Delta_{\mathcal{W}_R}^{it} = U(v_R(2\pi t)), \quad (6.1)$$

where Θ is the PCT-operator associated to the Wightman field, $v_R(t)$, $t \in \mathbb{R}$, is the one-parameter subgroup of boosts leaving the wedge \mathcal{W}_R invariant, and $R_1(\pi)$ is the rotation through the angle π about the 1-axis, which is perpendicular to the edge of \mathcal{W}_R (similar relations are valid for the modular objects of any wedge). Hence,

$$J_{\mathcal{W}_R} \mathcal{A}(\mathcal{O}) J_{\mathcal{W}_R} = \mathcal{A}(\theta_{\mathcal{W}_R} \mathcal{O}), \Delta_{\mathcal{W}_R}^{it} \mathcal{A}(\mathcal{O}) \Delta_{\mathcal{W}_R}^{-it} = \mathcal{A}(v_R(2\pi t) \mathcal{O}),$$

for all \mathcal{O} . Note for later use that $\Delta_{\mathcal{W}_R}^{it} = U(v_R(2\pi t)) = e^{i2\pi t K_1}$ entails

$$\Delta_{\mathcal{W}_R}^{1/2} = e^{\pi K_1}. \quad (6.2)$$

Though some of the significance of their insight will become clear below, we refer the reader to [186] for further development and references.

A fairly general strategy [17, 36] to construct a quantum field theory algebraically is to construct a unitary representation $U(\mathcal{P}_+^\uparrow)$ (the representation would be fixed by an analysis of particle masses, types and multiplicities in scattering

²⁹Roughly speaking, the bounded functions of the field operators $\phi(f)$ smeared with test functions f having support contained in the spacetime region \mathcal{O} are contained in the algebra $\mathcal{A}(\mathcal{O})$.

experiments) satisfying the spectrum condition³⁰ and for a fixed wedge \mathcal{W}_0 exhibit an algebra \mathfrak{G} which satisfies the *consistency conditions*:

(a) $U(\lambda) \mathfrak{G} U(\lambda)^{-1} \subset \mathfrak{G}$, whenever $\lambda \mathcal{W}_0 \subset \mathcal{W}_0$ for $\lambda \in \mathcal{P}_+^\uparrow$.

(b) $U(\lambda') \mathfrak{G} U(\lambda')^{-1} \subset \mathfrak{G}'$, whenever $\lambda' \mathcal{W}_0 \subset \mathcal{W}_0'$ for $\lambda' \in \mathcal{P}_+^\uparrow$, where a primed algebra denotes the commutant of the algebra and a primed wedge denotes the complementary wedge, *i.e.* the maximal wedge which is spacelike separated from the given wedge.

Then setting $\mathcal{A}(\mathcal{W}) \equiv U(\lambda) \mathfrak{G} U(\lambda)^{-1}$, where $\lambda \in \mathcal{P}_+^\uparrow$ is such that $\mathcal{W} = \lambda \mathcal{W}_0$ for any $\mathcal{W} \in \mathcal{W}$, one obtains an “algebra of observables” for each wedge region. For any other (convex) causally complete spacetime region \mathcal{O} one can then define $\mathcal{A}(\mathcal{O})$ as the intersection of all wedge algebras $\mathcal{A}(\mathcal{W})$ such that $\mathcal{O} \subset \mathcal{W}$. The resulting net $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O} \in \mathcal{R}}$ satisfies the conditions of isotony, relativistic covariance and Einstein causality. Conversely, any asymptotically complete quantum field theory fixes an algebra \mathfrak{G} satisfying the consistency conditions. One may therefore view such pairs (U, \mathfrak{G}) as germs of quantum field models. However, at present one does not have a general dynamical principle by which the algebras \mathfrak{G} may be selected, given the representation U . In the following, we briefly outline what may be seen as various ways of arriving at such pairs.

6.1 Modular localization

Though there have been many constructions of free quantum fields using various techniques, Bisognano and Wichmann’s breakthrough motivated Brunetti, Guido and Longo [29] to find what may be viewed as an intrinsic construction which we discuss next. We restrict our discussion to $d = 4$; the case $d = 3$ is more complex, as anyons are admitted [145]. One of the manifold aspects of the subtle notion of “particle” in QFT arises from Wigner’s famous classification of the irreducible representations of the (covering group of the) Poincaré group [195], in which each such representation is uniquely determined (up to a natural equivalence) by two numbers that can be interpreted as the mass and the spin of the particle (if the mass is strictly positive; when the mass is zero, there is the additional case of “continuous spin”). So presented with the task to construct a quantum field describing free particles of given mass and spin, one begins with the corresponding irreducible representation $U_1(\mathcal{P}_+)$ of the proper Poincaré group on a Hilbert space \mathcal{H}_1 . From this data one constructs a representation $U(\mathcal{P}_+)$ on \mathcal{H} , the bosonic Fock space with one-particle space \mathcal{H}_1 . Writing the representation of the boost subgroup leaving \mathcal{W}_R invariant in terms of its self-adjoint generator, $U_1(v_R(t)) = e^{itK_1}$, one *defines*, in light of (6.1) and (6.2) above,

$$\Delta_{\mathcal{W}_R}^{1/2} \equiv e^{\pi K_1}, \quad J_{\mathcal{W}_R} \equiv U_1(\theta_{\mathcal{W}_R})$$

³⁰Indeed, due to the fact that these wedge algebras are generally type III factors and therefore unitarily equivalent, it would suffice to do so on Fock space.

Motivated by Tomita–Takesaki theory one then defines

$$S_{\mathcal{W}_R} \equiv U_1(\theta_{\mathcal{W}_R})e^{\pi K_1} = J_{\mathcal{W}_R}\Delta_{\mathcal{W}_R}^{1/2}$$

and identifies the corresponding real subspace of invariant vectors in \mathcal{H}_1

$$\mathcal{K}_{\mathcal{W}_R} \equiv \{f \in D(S_{\mathcal{W}_R}) \mid S_{\mathcal{W}_R}f = f\}$$

(similarly for all $\mathcal{W} \in \mathbf{W}$). On the exponential (or *coherent*) vectors

$$e^h \doteq \oplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}} h^{\otimes n}, \quad h \in \mathcal{H}_1,$$

which are total in \mathcal{H} , one defines unitary operators $V(f)$, $f \in \mathcal{H}_1$, by

$$V(f)e^0 \doteq e^{-\frac{1}{4}\|f\|^2} e^{\frac{i}{\sqrt{2}}f}, \quad f \in \mathcal{H}_1,$$

$$V(f)V(g) \doteq e^{-\frac{i}{2}\text{Im}\langle f, g \rangle} V(f+g), \quad f, g \in \mathcal{H}_1.$$

Now let $\mathcal{A}(\mathcal{W}_R)$ denote the von Neumann algebra generated by $\{V(f) \mid f \in \mathcal{K}_{\mathcal{W}_R}\}$. Brunetti, Guido and Longo showed that if $U_1(\mathcal{P}_+)$ satisfies the spectrum condition, then $(U(\mathcal{P}_+^\uparrow), \mathcal{A}(\mathcal{W}_R))$ satisfies the consistency conditions and therefore determines a local, Poincaré covariant net $\{\mathcal{A}(\mathcal{O})\}$. In the case of finite spin, the resulting net coincides with the net of local algebras associated with the corresponding free field; in the special case of massless, “infinite (or continuous) spin” representations U_1 , this provides the first construction of a quantum field model covariant under an infinite spin representation. Indeed, there is no quantum field of the standard type associated with the net in this case, since the Fock vacuum vector $\Omega = e^0$ is not cyclic for algebras $\mathcal{A}(\mathcal{O})$ when \mathcal{O} is bounded; instead, these nets are generated by string-localized fields [145]. However, for any choice of positive energy representation U_1 the vector Ω is cyclic for $\mathcal{A}(\mathcal{W})$, for any wedge \mathcal{W} , and for $\mathcal{A}(\mathcal{C})$, for any spacelike cone \mathcal{C} . In the case that $U_1(\mathcal{P}_+)$ is irreducible with finite spin, Ω is also cyclic for $\mathcal{A}(\mathcal{O})$, for any double cone \mathcal{O} .

6.2 Models with nontrivial factorizing S–matrices

A classic problem of obvious physical importance is the so-called *inverse scattering problem*, which in the context of QFT is: given the scattering matrix S (determined, in principle, by the measured scattering data in an experiment), does there exist a quantum field model whose S–matrix is the specified operator S ? As this is an enormously difficult problem, in order to make the problem more manageable workers in the field have focussed their attention on the special case of factorizing S–matrices in two spacetime dimensions; this is the relatively simple situation where all scattering processes reduce to (suitable combinations of) two-body scattering, so that specification of the two-body scattering amplitude completely determines the S–matrix. There is, therefore, no particle production in such models. The primary effort in this direction has been made in the context

of the form factor program [121, 176, 199], in which local quantum field operators associated with the quantum field purportedly having the prescribed scattering behavior are expressed in terms of a certain algebra, the Zamalodchikov algebra. Rigorous formulas for matrix elements of local operators between scattering states have been obtained, but the computation of products of local operators at different spacetime points is not under mathematical control, because infinite sums over intermediate states are involved. Hence, the Wightman axioms have not been verified in such models, apart from simple cases.

However, Schroer [167, 168] realized that certain field operators in the Zamalodchikov algebra can be interpreted as being localized in wedges. These wedge-localized but nonlocal quantum field operators that create covariant one-particle states out of the vacuum are now called *polarization-free generators*. It was subsequently shown by Borchers, Buchholz and Schroer [24] that in more than two spacetime dimensions the existence of (tempered) polarization-free generators defined on a translation-invariant, common dense domain (tacitly assumed in [167, 168]) entails the triviality of the associated S-matrix. However, for constructing quantum field models with nontrivial scattering in two dimensional Minkowski space the idea is quite fruitful.

In the special case of these factorizing S-matrix models the S-matrix is determined by a single function S_2 through the relation

$$(S\Psi)_n(\theta_1, \dots, \theta_n) = \left[\prod_{1 \leq l < k \leq n} S_2(|\theta_l - \theta_k|) \right] \Psi_n(\theta_1, \dots, \theta_n),$$

which defines its action on a general n -particle wave function in the Hilbert space described below, where $\theta_i, i = 1, \dots, n$, are the rapidities of the scattered particles. Lechner [133–135] considers a large class of two body scattering functions $S_2(\theta)$ satisfying conditions arising from the requirements of unitarity, crossing symmetry and hermitian analyticity of the corresponding S-matrix and uses them to define a concrete representation of the Zamalodchikov algebra. With $\mathcal{H}_1 = L^2(\mathbb{R})$, let \mathcal{H} be the S_2 -symmetrized Fock space, where the n -particle wave functions satisfy

$$\Psi_n(\theta_1, \dots, \theta_{i+1}, \theta_i, \dots, \theta_n) = S_2(\theta_i - \theta_{i+1}) \Psi_n(\theta_1, \dots, \theta_i, \theta_{i+1}, \dots, \theta_n).$$

For the special, and admissible, scattering functions $S_2 = 1, -1$, \mathcal{H} is the standard boson, respectively fermion, Fock space. \mathcal{H} admits a canonical, strongly continuous unitary representation $U(\mathcal{P}_+^\uparrow)$ of the Poincaré group satisfying the spectrum condition. On \mathcal{H} also act creation, resp. annihilation, operators Z^\dagger, Z , satisfying the Fadeev–Zamolodchikov relations:

$$Z^\dagger(\theta) Z^\dagger(\theta') = S_2(\theta - \theta') Z^\dagger(\theta') Z^\dagger(\theta)$$

(similarly for Z) and

$$Z(\theta) Z^\dagger(\theta') = S_2(\theta' - \theta) Z^\dagger(\theta') Z(\theta) + \delta(\theta - \theta') \cdot \mathbb{1}$$

In terms of these one defines a quantum field operator

$$\phi(f) \doteq Z^\dagger(f_+) + Z(f_-), \quad f \in \mathcal{S}(\mathbb{R}^2),$$

where

$$f_\pm(\theta) = \frac{1}{2\pi} \int f(x) e^{\pm i p(\theta)x} dx$$

and $p(\theta) = m(\cosh \theta, \sinh \theta)$, where $m > 0$ is the mass.

Though the field $\phi(x)$ is covariant under the action of $U(\mathcal{P}_+^\dagger)$, is densely defined, and is a distributional solution of the Klein-Gordon equation of mass m , it does not satisfy Einstein causality unless $S_2 = 1$, in which case $\phi(x)$ is the usual free scalar Bose field. However, as observed by Schroer, when smeared with test functions having support in a wedge, they are polarization-free generators.

In two dimensional Minkowski space the set of wedges decomposes into two components, one is the set of all translates of \mathcal{W}_R and the other is the set of all translates of $\mathcal{W}_L \equiv \mathcal{W}_R'$. Defining $\mathcal{A}(\mathcal{W}_R)$ to be the von Neumann algebra generated by $\{e^{i\phi(f)} \mid \text{supp}(f) \subset \mathcal{W}_R\}$, $\mathcal{A}(\mathcal{W}_L) \equiv \mathcal{A}(\mathcal{W}_R)'$, and $\mathcal{A}(\mathcal{W}) \equiv U(x)\mathcal{A}(\mathcal{W}_R)U(x)^{-1}$ if $\mathcal{W} = \mathcal{W}_R + x$ (similarly for $\mathcal{W} = \mathcal{W}_L + x$), then $\{\mathcal{A}(\mathcal{W})\}_{\mathcal{W} \in \mathcal{W}}$ satisfies the HAK axioms for the restricted collection $\mathcal{R} = \mathcal{W}$. The Fock vacuum vector Ω is cyclic for the wedge algebras and the corresponding modular objects coincide with those found in the Bisognano–Wichmann setting.

In two spacetime dimensions double cones can be defined simply as the intersection of two suitable wedges: $\mathcal{O} = \mathcal{W}_1 \cap \mathcal{W}_2'$ with $\mathcal{W}_2 \subset \mathcal{W}_1$. Lechner showed that if one defines the corresponding observable algebra to be

$$\mathcal{A}(\mathcal{O}) \equiv \mathcal{A}(\mathcal{W}_1) \cap \mathcal{A}(\mathcal{W}_2)' (= \mathcal{A}(\mathcal{W}_1) \cap \mathcal{A}(\mathcal{W}_2')),$$

one finds that for a large class of S_2 (*i.e.* for a large class of models of the type we are discussing), Ω is cyclic and separating for all double cone algebras and the double cone algebras satisfy Einstein causality. Moreover, the Haag–Ruelle scattering theory can then be applied to yield a scattering theory which is asymptotically complete and whose S-matrix coincides with the initially prescribed S-matrix [135]. These models therefore constitute a complete and satisfying solution to the inverse scattering problem for the stated class of S-matrices.

Note that the construction is implemented using easily constructed, but nonlocal fields to obtain local wedge algebras, of which suitable relative commutants provide algebras of observables localized in bounded spacetime regions. This sidesteps the usual process of first constructing local fields and then using them to obtain the local observable algebras. This is a significant simplification for the models in question, since arguments by Smirnov and Schroer [168, 176] and examples laboriously computed by McCoy *et alia* [144] indicate that the *local* fields in these models must be *infinite power series* in the (simple) nonlocal fields. By constructing the local algebras in the indicated manner, one is able to avoid controlling the infinite expansion that yields the local fields and yet still arrive at the desired quantities of physical relevance.

The special case of $S_2 = -1$ was studied by Buchholz and Summers [35] for $d \geq 2$, and it was shown that the model is maximally nonlocal in a certain specific quantitative sense. For $d = 2$ it was shown that there are two associated local nets admitting an asymptotically complete scattering theory, one describing a fermion with trivial scattering and another describing a boson with $S = (-1)^{N(N-1)/2}$, where N is the number operator. In higher dimensions there exist string-localized, respectively brane-localized, operators which mutually commute at spacelike separation.

6.3 Deformations

Deformation techniques have long been used to provide quantized versions of classical theories (see *e.g.* [192]), but attempts to deform quantum field models have appeared fairly recently. Their initial impetus was provided by the desire to deform quantum fields on Minkowski space to arrive at quantum fields on Moyal space, or noncommutative Minkowski space. Though most of that work has not been mathematically rigorous, Grosse and Lechner [99] rigorously defined a deformation of the scalar massive free Bose field in Wightman's setting, which could then be interpreted either as a model on noncommutative Minkowski space or as a model on Minkowski space. Of particular interest is that the deformed field manifests nontrivial scattering (see below). Buchholz and Summers [36] then found a deformation of essentially any algebraic quantum field model which coincides with the deformation of Grosse and Lechner when restricted to the net associated with the mentioned free field. We discuss this deformation and the resulting models here. Though this procedure may be carried out for $d \geq 2$, we restrict our discussion to $d = 4$.

Let $(\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O} \in \mathcal{R}}, \mathcal{H}, U, \Omega)$ satisfy the conditions stated in Section 1; let $U(x) = e^{iPx}$, $x \in \mathbb{R}^4$, and $P = \int p \, dE(p)$ be the joint spectral decomposition of the generators of $U(\mathbb{R}^4)$. The support of the projection-valued measure E is $\overline{V_+}$, by the spectrum condition. Let M be a bounded operator and, for convenience, let

$$\alpha_p(M) \equiv e^{iPp} M e^{-iPp} \in \mathcal{B}(\mathcal{H}), p \in \mathbb{R}^4.$$

The matrices

$$Q \equiv \begin{pmatrix} 0 & \zeta & 0 & 0 \\ \zeta & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta \\ 0 & 0 & -\eta & 0 \end{pmatrix}$$

for fixed $\zeta \geq 0$, $\eta \in \mathbb{R}$, are uniquely distinguished by the following properties [99]:

- (i) $Q \overline{V_+} \subset \mathcal{W}_R$.
- (ii) If $\lambda = (\Lambda, x) \in \mathcal{P}_+^\uparrow$ satisfies $\lambda \mathcal{W}_R \subset \mathcal{W}_R$, then $\Lambda Q \Lambda^T = Q$.
- (iii) If $\lambda = (\Lambda, x) \in \mathcal{P}_+^\uparrow$ satisfies $\lambda \mathcal{W}_R \subset \mathcal{W}'_R$, then $\Lambda Q \Lambda^T = -Q$.

The *warped convolution* of M is given by the formal expression³¹

$$M_Q = \int dE(p) \alpha_{Qp}(M), \quad M \in \mathcal{B}(\mathcal{H}). \quad (6.3)$$

A detailed explication of this unusual operator valued integral is given in [37]. One finds that M_Q is also a bounded operator on \mathcal{H} , so one may further define the deformed algebra $\mathcal{A}_Q(\mathcal{W}_R)$ corresponding to the algebra $\mathcal{A}(\mathcal{W}_R)$ to be the von Neumann algebra generated by $\{A_Q \mid A \in \mathcal{A}(\mathcal{W}_R)\}$. Then $(U, \mathcal{A}_Q(\mathcal{W}_R))$ satisfies the consistency conditions [36, 37] and therefore generates, as above, a net satisfying the Haag-Araki-Kastler axioms. Although the original net and the deformed net are not isomorphic, the modular objects corresponding to the pair $(\mathcal{A}(\mathcal{W}_R), \Omega)$ coincide with those corresponding to $(\mathcal{A}_Q(\mathcal{W}_R), \Omega)$ [37].

There are indications that the deformed algebras corresponding to bounded spacetime regions \mathcal{O} may be trivial, *i.e.* are multiples of the identity operator, so one may not be able to formulate a full scattering theory for the deformed net. However, two-body scattering for the deformed model is well defined [36]. The relations between the two-body scattering states in the original and in the deformed theory are most transparent if one uses improper single particle states of sharp momentum $p = (\sqrt{p^2 + m^2}, \mathbf{p})$, $q = (\sqrt{q^2 + m^2}, \mathbf{q})$. There one has [36]

$$\begin{aligned} |p \otimes_Q q\rangle^{\text{in}} &= e^{i|pQq|} |p \otimes q\rangle^{\text{in}} \\ |p \otimes_Q q\rangle^{\text{out}} &= e^{-i|pQq|} |p \otimes q\rangle^{\text{out}}. \end{aligned}$$

The scattering states in the deformed theory depend on the matrix Q through the choice of the wedge \mathcal{W}_R and thus break the Lorentz symmetry in $d > 2$ dimensions. This can be understood if one interprets the deformed theory as living on noncommutative Minkowski space, where the Lorentz symmetry is broken.

The kernels of the elastic scattering matrices in the deformed and undeformed theory are related by

$${}^{\text{out}}\langle p \otimes_Q q | p' \otimes_Q q' \rangle^{\text{in}} = e^{i|pQq| + i|p'Qq'|} {}^{\text{out}}\langle p \otimes q | p' \otimes q' \rangle^{\text{in}}.$$

Thus they differ, and even if the initial model has trivial scattering, the deformed theory does not. Hence, this deformation applied to the net of the scalar massive free field results in a mathematically rigorous quantum field model with nontrivial scattering, apparently the first such model in four spacetime dimensions.

Dybalski and Tanimoto [57] have shown that application of the warped convolution to any chiral conformal QFT in two spacetime dimensions results in a model which has Lorentz invariant nontrivial scattering and is asymptotically complete, the first example of a massless QFT having such properties.

These deformations were subsequently reinterpreted in the Wightman framework by Grosse and Lechner [100] in terms of a deformed product on the Borchers

³¹Strictly speaking, this is well defined for those bounded M which are smooth with respect to the action α_x , $x \in \mathbb{R}^4$ [37].

algebra associated with the polynomials in the field operators mentioned in Section 1. And, again in that framework, Lechner [136] has found a larger class of deformations which includes warped deformations as a special case. In four space-time dimensions, the resulting deformed theories have properties similar to those of the warped models; however, in two dimensions he showed that the models discussed in Section 6.2 are obtained from the free field by a deformation of this new type. Common to all of these deformed models so far is the absence of particle production.

7 Perturbative AQFT

As only briefly indicated in Section 1, the algebraic approach to QFT may comfortably consider local algebras of observables which are not algebras of bounded operators acting on a Hilbert space. In this section we discuss briefly an interesting example of this which goes well beyond the polynomial algebras of the Wightman approach. This work, carried out primarily by Brunetti, Dütsch and Fredenhagen, explicitly incorporates perturbative QFT into the framework of AQFT. Though one of the primary recent concerns of these authors is to formulate perturbative AQFT in a manner independent of any background space-time, we shall present an early version of their formulation, since we are concerned here with relativistic QFT on Minkowski space. This will simplify the discussion somewhat. For the same reason, we shall not try to describe the steps taken by the authors to incorporate more general interactions and shall, instead, sketch an early version of this approach.

As with Borchers algebras, the connection between an abstract (complex, unital) $*$ -algebra \mathcal{A} and operators acting on a Hilbert space \mathcal{H} (where the probabilistic interpretation familiar from quantum theory applies) is implemented by a *representation*, a map π from \mathcal{A} into the set of (densely defined) operators acting on \mathcal{H} which satisfies $\pi(cA+B) = c\pi(A) + \pi(B)$, $\pi(AB) = \pi(A)\pi(B)$ and $\pi(A^*) = \pi(A)^*$, for all $A, B \in \mathcal{A}$ and $c \in \mathbb{C}$. If one finds a *state* ω on \mathcal{A} (a linear map $\omega : \mathcal{A} \rightarrow \mathbb{C}$ such that $\omega(A^*A) \geq 0$ for all $A \in \mathcal{A}$ and $\omega(I) = 1$, where I is the unit in \mathcal{A}), then such a Hilbert space \mathcal{H} and representation π , called the GNS representation associated with ω , can be canonically constructed from the data (\mathcal{A}, ω) . In this space \mathcal{H} there is a unit vector Ω such that $\omega(A) = \langle \Omega, \pi(A)\Omega \rangle$ for all $A \in \mathcal{A}$.

In perturbative AQFT (as in some forms of deformation quantization [192]) the basic objects are understood as formal power series. Consider the set $\mathbb{C}[[t]]$ of all complex sequences $\{c_0, c_1, \dots\}$, where to each such sequence corresponds a formal power series in the formal parameter t

$$c = \sum_{n=0}^{\infty} c_n t^n.$$

On this set addition is defined termwise and a product is defined using Cauchy's

formula:

$$ab = \left(\sum_{n=0}^{\infty} a_n t^n \right) \left(\sum_{n=0}^{\infty} b_n t^n \right) \equiv \sum_{n=0}^{\infty} \left(\sum_{m=0}^n a_m b_{n-m} \right) t^n.$$

With these operations, $\mathbb{C}[[t]]$ is an associative, commutative ring with unit. If \mathcal{A} is a complex algebra, then $\mathcal{A}[[t]]$ is defined as the set of all sequences with entries in \mathcal{A} and, again, may be thought of as the set of all formal power series with coefficients in \mathcal{A} . With addition, multiplication and scalar multiplication (with scalars in $\mathbb{C}[[t]]$) defined similarly, $\mathcal{A}[[t]]$ is an algebra over $\mathbb{C}[[t]]$.

Perturbative AQFT is based upon the causal perturbation theory developed primarily by Stückelberg, Bogoliubov, Shirkov, Epstein and Glaser. This theory cannot be described here (*cf. e.g.* [59]), but it is a mathematically rigorous renormalization theory for quantum field theoretical perturbation series which preserves and utilizes in an essential manner “locality” properties, in the double sense commonly understood in AQFT — localization properties coupled with Einstein causality. In the papers discussed below the ultraviolet problems with the formal series representations of the observables are handled by techniques from causal perturbation theory. These techniques are not described here.

In the Fock space of the free scalar massive Bose field, consider the cutoff interaction Hamiltonian

$$H_I(t) = - \int g(t, \mathbf{x}) A(t, \mathbf{x}) d^3x,$$

with g an infinitely differentiable function with compact support and A a (derivative of a) Wick polynomial. The corresponding time evolution operator from time $-\tau$ to time τ , where $\tau > 0$ is sufficiently large that $(-\tau, \tau) \times \mathbb{R}^3$ contains the support of g , is formally given by the Dyson series

$$S(g) = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int T(A(x_1) \cdots A(x_n)) g(x_1) \cdots g(x_n) dx_1 \cdots dx_n, \quad (7.1)$$

where the time ordered products $T(A(x_1) \cdots A(x_n))$ are operator valued distributions on \mathcal{D} satisfying

$$T(A(x_1) \cdots A(x_n)) = T(A(x_1) \cdots A(x_k)) T(A(x_{k+1}) \cdots A(x_n))$$

whenever all of x_{k+1}, \dots, x_n do not lie in the forward lightcone of any of the points x_1, \dots, x_k . Supplying the test function g with a factor (coupling constant) λ , the local S-matrix (7.1) is to be understood as an element of $\mathcal{A}[[\lambda]]$, where the algebra \mathcal{A} is a natural extension of the Borchers class of the free field. $S(g)$ has an inverse in $\mathcal{A}[[\lambda]]$ of the form (7.1) with i replaced by $-i$ and the time ordered products are replaced by “antichronological” products

$$\overline{T}(A(x_1) \cdots A(x_n)) \equiv \sum_{P \in \mathcal{P}(\{1, \dots, n\})} (-1)^{|P|+n} \prod_{p \in P} T(A(x_i), i \in P),$$

where $\mathcal{P}(\{1, \dots, n\})$ is the set of all ordered partitions of $\{1, \dots, n\}$ and $|P|$ is the number of subsets in P . The “antichronological” products satisfy anticausal factorization. As observed by Il’in and Slavnov [112], the local S–matrices satisfy

$$S(f + g + h) = S(f + g)S(g)^{-1}S(g + h), \quad (7.2)$$

whenever the support of h is disjoint from the causal future of the support of f (independently of g). Further solutions of (7.2) are obtained by introducing the relative S–matrices

$$S_g(f) \equiv S(g)^{-1}S(g + f),$$

which also satisfy local commutation relations $[S_g(h), S_g(f)] = 0$ whenever the supports of h and f are spacelike separated.

Local algebras of observables are then introduced by defining $\mathcal{A}_g(\mathcal{O})$ to be the $*$ –algebra generated by the relative S–matrices $S_g(h)$ whose test functions h have support contained in \mathcal{O} . If $g = g'$ in a neighborhood of a causally closed region containing \mathcal{O} , then there exists a unitary $V \in \mathcal{A}[[\lambda]]$ such that $VS_g(h)V^{-1} = S_{g'}(h)$ for all test functions with support in \mathcal{O} . Capitalizing upon this fact, Dütsch and Fredenhagen [56] construct for a given interaction Lagrangian \mathcal{L} (here a polynomial in the field or derivatives of the field) a net of such local $*$ –algebras which satisfies Einstein causality and is covariant under a natural adjoint action of the usual unitary representation of the Poincaré group defined on the initial Fock space.

Dütsch and Fredenhagen [55] have shown that this approach can be employed also for QED. Because the basic quantities are local and independent of the test function g in the sense indicated above, no reference need be made to the adiabatic limit. The incompatibility in QED between gauge invariance, locality and positive definite inner products on the state space when the unobservable gauge potentials and charge carrying fields are employed is addressed by a local construction of the observables and of the physical Hilbert space in which the observables are faithfully represented (once again as formal power series of unbounded operators).

More recently, Brunetti, Dütsch and Fredenhagen [30] have refined this approach to allow the treatment of low dimensional theories and non-polynomial interactions. In addition, they studied in this framework three of the various approaches to renormalization group ideas that are in use among theoretical physicists and have established their mutual logical relations. They obtain an algebraic form of the Callan–Symanzik equation and compute the β function in the ϕ_4^4 and ϕ_6^3 interactions in their framework, finding perfect agreement between their results and those found by heuristic methods.

Although this work does not provide quantum field models satisfying the HAK or Wightman axioms, it falls comfortably within the framework of AQFT, since the primary objects are, again, nets of local $*$ –algebras generated by observables which are Poincaré covariant and satisfy Einstein causality and, again, the work is carried out with complete mathematical rigor. However, when these authors speak of representations in Hilbert spaces, the Hilbert spaces are vector spaces over the field $\mathbb{C}[[\lambda]]$, not over \mathbb{C} . So taking expectations of observables in states in this

approach results in a formal complex power series, not a complex number. Hence, in order to make the connection to experiments one must deliberately consider a partial sum of this series, *i.e.* consider the perturbation series only to a finite order, as is done in heuristic QFT. Since these series are not convergent, one is returned to the question “Is there an exact model?”

8 Outlook

It is evident that the efforts of the constructive quantum field theorists have been crowned with many successes. They have constructed superrenormalizable models, renormalizable models and even nonrenormalizable models, as well as models which fall outside of that classification scheme since they apparently do not correspond to some classical Lagrangian. And they have found means to extract rigorously from these models physically and mathematically crucial information. In many of these models the HAK and the Wightman axioms have been verified. In the models constructed to this point, the intuitions/hopes of the quantum field theorists have been largely confirmed. However, local gauge theories such as quantum electrodynamics, quantum chromodynamics and the Standard Model — precisely the theories whose approximations of various kinds are used in a central manner by elementary particle theorists and cosmologists — remain to be constructed. These models present significant mathematical and conceptual challenges to all those who are not satisfied with *ad hoc* and essentially instrumentalist computation techniques.

Why haven’t these models of greatest physical interest been constructed yet (in any mathematically rigorous sense which preserves the basic principles constantly evoked in heuristic QFT and does not satisfy itself with an approximation)? Certainly, one can point to the practical fact that only a few dozen people have worked in CQFT. This should be compared with the many hundreds working in string theory and the thousands who have worked in elementary particle physics. Progress is necessarily slow if only a few are working on extremely difficult problems.³² It may well be that patiently proceeding along the lines indicated above and steadily improving the technical tools employed will ultimately yield the desired rigorous constructions. It may also be the case that a completely new approach is required, though remaining within the CQFT program as described in Section 1, something whose essential novelty is analogous to the differences between the approaches in Section 2, 3, 5 and 6.

It may even be the case that, as Gurau, Magnen and Rivasseau have written [103], “perhaps axiomatization of QFT might have been premature”; in other words, perhaps the Wightman and HAK axioms do not provide the proper mathematical framework for QED, QCD, SM, even though, as the constructive quantum

³²Indeed, many workers in CQFT have chosen to take the methods developed for the purposes of CQFT and to apply them instead to problems in statistical mechanics, many body physics and solid state physics, where progress has been much easier.

field theorists have so convincingly demonstrated, that framework is quite suitable for so many models of such varying types and, as the algebraic quantum field theorists have just as convincingly demonstrated, that framework is flexible and powerful when dealing with the conceptual and mathematical problems in QFT which go *beyond* mathematical existence. But it is possible that the mathematically and conceptually essential core of a rigorous formulation of QFT that can include the missing models lies somewhere else. Certainly, there are presently many attempts to understand aspects of QFT from the perspective of mathematical ideas which are quite unexpected when seen from the vantage point of current QFT and even from the vantage point of quantum theory itself, as rigorously formulated by von Neumann and many others. These speculations, as suggestive as some may be, are currently beyond the scope of this article.

References

- [1] M. Aizenman, Geometric analysis of ϕ_4^4 fields and Ising models, *Commun. Math. Phys.*, **86**, 1–48 (1982).
- [2] S. Albeverio and R. Hoegh-Krohn, The Wightman axioms and the mass gap for strong interactions of exponential type in two-dimensional space-time, *J. Funct. Anal.*, **16**, 39–82 (1974).
- [3] S. Albeverio, G. Gallavotti and R. Hoegh-Krohn, Some results for the exponential interaction in two or more dimensions, *Commun. Math. Phys.*, **70**, 187–192 (1979).
- [4] S. Albeverio, H. Gottschalk and J.-L. Wu, Models of local relativistic quantum fields with indefinite metric (in all dimensions), *Commun. Math. Phys.*, **184**, 509–531 (1997).
- [5] S. Albeverio and H. Gottschalk, Scattering theory for quantum fields with indefinite metric, *Commun. Math. Phys.*, **216**, 491–513 (2001).
- [6] M. Anshelevich and A.N. Sengupta, Quantum free Yang–Mills on the plane, preprint arXiv:1106.2107.
- [7] H. Araki, On the algebra of all local observables, *Prog. Theor. Phys.*, **32**, 844–854 (1964).
- [8] H. Araki, Von Neumann algebras of local observables for free scalar field, *J. Math. Phys.*, **5**, 1–13 (1964).
- [9] H. Araki, *Mathematical Theory of Quantum Fields* (Oxford University Press, Oxford) 1999.
- [10] A. Ashtekar and J. Lewandowski, Representation theory of analytic holonomy C^* algebras, in: *Knots and Quantum Gravity*, ed. by J. Baez (Oxford University Press, Oxford) 1994, pp. 21–61.
- [11] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourao and T. Thiemann, $SU(N)$ Quantum Yang–Mills theory in two dimensions: A complete solution, *J. Math. Phys.*, **38**, 5453–5472 (1997).
- [12] J.C. Baez, I.E. Segal and Z. Zhou, *Introduction to Algebraic and Constructive Quantum Field Theory* (Princeton University Press, Princeton) 1992.

- [13] T. Balaban and K. Gawedzki, A Low temperature expansion for the pseudoscalar Yukawa model of quantum field in two space-time dimensions, *Ann. Inst. Henri Poincaré*, **36**, 271–400 (1982).
- [14] T. Balaban, (Higgs)_{2,3} quantum fields in a finite volume. III. Renormalization, *Commun. Math. Phys.*, **88**, 411–445 (1983).
- [15] T. Balaban, J.Z. Imbrie and A. Jaffe, Effective action and cluster properties of the abelian Higgs model, *Commun. Math. Phys.*, **114**, 257–315 (1988).
- [16] T. Balaban, Large field renormalization. II. Localization, exponentiation, and bounds for the R operation, *Commun. Math. Phys.*, **122**, 355–392 (1989).
- [17] H. Baumgärtel and M. Wollenberg, *Causal Nets of Operator Algebras* (Akademie Verlag, Berlin) 1992.
- [18] J. B  llisard, J. Fr  hlich and B. Gidas, Soliton mass and surface tension in the $(\lambda|\phi|^4)_2$ quantum field model, *Commun. Math. Phys.*, **60**, 37–72 (1978).
- [19] G. Benfatto, P. Falco and V. Mastropietro, Functional integral construction of the massive Thirring model: Verification of axioms and massless limit, *Commun. Math. Phys.*, **273**, 67–118 (2007).
- [20] G. Benfatto, P. Falco and V. Mastropietro, Massless Sine–Gordon and massive Thirring models: Proof of Coleman’s equivalence, *Commun. Math. Phys.*, **285**, 713–762 (2009).
- [21] J.J. Bisognano and E.H. Wichmann, On the duality condition for a Hermitian scalar field, *J. Math. Phys.*, **16**, 985–1007 (1975).
- [22] H.J. Borchers, On the structure of the algebra of field operators, *Nuovo Cim.*, **24**, 214–236 (1962).
- [23] H.-J. Borchers and J. Yngvason, From quantum fields to local von Neumann algebras, *Rev. Math. Phys.*, **Special Issue**, 15–47 (1992).
- [24] H.-J. Borchers, D. Buchholz and B. Schroer, Polarization–free generators and the S–matrix, *Commun. Math. Phys.*, **219**, 125–140 (2001).
- [25] V.S. Borkar, R.T. Chari and S.K. Mitter, Stochastic quantization of field theory in finite and infinite volume, *J. Funct. Anal.*, **81**, 184–206 (1988).
- [26] O. Bratteli and D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, Volume 1 (Springer Verlag, Berlin) 1979, Volume 2 (Springer Verlag, Berlin) 1981.
- [27] J. Bros and D. Iagolnitzer, 2–Particle asymptotic completeness and bound states in weakly coupled quantum field theories, *Commun. Math. Phys.*, **119**, 331–351 (1988).
- [28] J. Bros and D. Buchholz, Towards a relativistic KMS condition, *Nucl. Phys. B*, **429**, 291–318 (1994).
- [29] R. Brunetti, D. Guido and R. Longo, Modular localization and Wigner particles, *Rev. Math. Phys.*, **14**, 759–785 (2002).
- [30] R. Brunetti, M. D  tsch and K. Fredenhagen, Perturbative algebraic quantum field theory and the renormalization groups, *Adv. Theor. Math. Phys.*, **13**, 1541–1599 (2009).

- [31] D.C. Brydges, J. Fröhlich and E. Seiler, On the construction of quantized gauge fields: III. The two-dimensional abelian Higgs model without cutoffs, *Commun. Math. Phys.*, **79**, 353–399 (1981).
- [32] D. Brydges, J. Dimock and T.R. Hurd, The short distance behavior of $(\phi^4)_3$, *Commun. Math. Phys.*, **172**, 143–186 (1995).
- [33] D. Buchholz, The physical state space of quantum electrodynamics, *Commun. Math. Phys.*, **85**, 49–71 (1982).
- [34] D. Buchholz, On quantum fields which generate local algebras, *J. Math. Phys.*, **31**, 1839–1846 (1990).
- [35] D. Buchholz and S.J. Summers, String- and brane-localized causal fields in a strongly nonlocal model, *J. Phys. A: Math. Theor.*, **40**, 2147–2163 (2007).
- [36] D. Buchholz and S.J. Summers, Warped convolutions: A novel tool in the construction of quantum field theories, in: *Quantum Field Theory and Beyond*, edited by E. Seiler and K. Sibold (World Scientific, Singapore), 2008, pp. 107–121.
- [37] D. Buchholz, G. Lechner and S.J. Summers, Warped convolutions, Rieffel deformations and the construction of quantum field theories, *Commun. Math. Phys.*, **304**, 95–123 (2011).
- [38] C. Burnap, Isolated one particle states in boson quantum field theory models, *Ann. Phys.*, **104**, 184–196 (1977).
- [39] C. de Calan, Construction of the Gross–Neveu model in dimension 3, in: *Constructive Physics (Palaiseau, 1994)*, Lecture Notes in Physics, **446** (Springer-Verlag, Heidelberg, Berlin, New York) 1995, pp. 149–159.
- [40] C. de Calan and V. Rivasseau, The perturbation series for ϕ_3^4 is divergent, *Commun. Math. Phys.*, **83**, 77–82 (1982).
- [41] A.L. Carey, S.N.M. Ruijsenaars and J.D. Wright, The massless Thirring model: Positivity of Klaiber’s n -point functions, *Commun. Math. Phys.*, **99**, 347–364 (1985).
- [42] S. Coleman, Quantum sine-Gordon equation as the massive Thirring model, *Phys. Rev. D*, **11**, 2088–2097 (1975).
- [43] A. Cooper and L. Rosen, The weakly coupled Yukawa₂ field theory: cluster expansion and Wightman axioms, *Trans. Am. Math. Soc.*, **234**, 1–88 (1977).
- [44] M. Davier and W.J. Marciano, The theoretical prediction for the muon anomalous magnetic moment, *Annu. Rev. Nucl. Part. Sci.*, **54**, 115–140 (2004).
- [45] J. Dimock and J.-P. Eckmann, On the bound state in weakly coupled $\lambda(\phi^6 - \phi^4)_2$, *Commun. Math. Phys.*, **51**, 41–54 (1976).
- [46] J. Dimock and J.-P. Eckmann, Spectral properties and bound-state scattering in weakly coupled $\lambda P(\phi)_2$ models, *Ann. Phys.*, **103**, 289–314 (1977).
- [47] J. Dimock and T.R. Hurd, Construction of the two-dimensional Sine–Gordon model for $\beta < 8\pi$, *Commun. Math. Phys.*, **156**, 547–580 (1993); corrected by the same authors in *Ann. Henri Poincaré*, **1**, 499–541 (2000).
- [48] M. Disertori and V. Rivasseau, Continuous constructive fermionic renormalization, *Ann. Henri Poincaré*, **1**, 1–57 (2000).

- [49] M. Donald, The classical field limit of $P(\phi)_2$ quantum field theory, *Commun. Math. Phys.*, **79**, 153–165 (1981).
- [50] W. Driessler and S.J. Summers, Central decomposition of Poincaré- invariant nets of local field algebras and absence of spontaneous breaking of the Lorentz group, *Ann. Inst. Henri Poincaré*, **43**, 147–166 (1985).
- [51] W. Driessler and S.J. Summers, On the decomposition of relativistic quantum field theories into pure phases, *Helv. Phys. Acta*, **59**, 331–348 (1986).
- [52] W. Driessler, S.J. Summers and E.H. Wichmann, On the connection between quantum fields and von Neumann algebras of local operators, *Commun. Math. Phys.*, **105**, 49–84 (1986).
- [53] B.K. Driver, Convergence of the $U(1)_4$ lattice gauge theory to its continuum limit, *Commun. Math. Phys.*, **110**, 479–501 (1987).
- [54] B.K. Driver, YM_2 : Continuum expectations, lattice convergence, and lassos, *Commun. Math. Phys.*, **123**, 575–616 (1989).
- [55] M. Dütsch and K. Fredenhagen, A local (perturbative) construction of observables in gauge theories: The example of QED, *Commun. Math. Phys.*, **203**, 71–105 (1999).
- [56] M. Dütsch and K. Fredenhagen, Algebraic quantum field theory, perturbation theory, and the loop expansion, *Commun. Math. Phys.*, **219**, 5–30 (2001).
- [57] W. Dybalski and Y. Tanimoto, Asymptotic completeness in a class of massless relativistic quantum field theories, *Commun. Math. Phys.*, **305**, 427–440 (2011).
- [58] F.J. Dyson, Divergence of perturbation theory in quantum electrodynamics, *Phys. Rev.*, **85**, 631–632 (1952).
- [59] H. Epstein and V. Glaser, The role of locality in perturbation theory, *Ann. Inst. Henri Poincaré*, **19**, 211–295 (1973).
- [60] L. Fassarella and B. Schroer, Wigner particle theory and local quantum physics, *J. Phys. A: Math. Gen.*, **35**, 9123–9164 (2002).
- [61] P. Federbush, A two-dimensional relativistic field theory, *Phys. Rev.*, **121**, 1247–1249 (1961).
- [62] P. Federbush, A Phase cell approach to Yang–Mills theory, V. Analysis of a chunk, *Commun. Math. Phys.*, **127**, 433–457 (1990).
- [63] J.S. Feldman and K. Osterwalder, The Wightman axioms and the mass gap for weakly coupled ϕ_3^4 quantum field theories, *Ann. Phys.*, **97**, 80–135 (1976).
- [64] J.S. Feldman and R. Raczka, The relativistic field equation of the $\lambda\phi_3^4$ quantum field theory, *Ann. Phys.*, **108**, 212–229 (1977).
- [65] J.S. Feldman, J. Magnen, V. Rivasseau and R. Sénéor, A Renormalizable field theory: The massive Gross–Neveu model in two dimensions, *Commun. Math. Phys.*, **103**, 67–103 (1986).
- [66] J.S. Feldman, T. Hurd, L. Rosen and J.D. Wright, *QED: A Proof of Renormalizability*, Lecture Notes in Physics, **312** (Springer-Verlag, Berlin, Heidelberg, New York) 1988.

- [67] R. Fernández, J. Fröhlich and A. Sokal, *Random Walks, Critical Phenomena, and Triviality in Quantum Field Theory* (Springer Verlag, New York) 1992.
- [68] C. Fleischhack, A new type of loop independence and $SU(N)$ quantum Yang–Mills theory in two dimensions, *J. Math. Phys.*, **41**, 76–102 (2000).
- [69] K. Fredenhagen and J. Hertel, Local algebras of observables and pointlike localized fields, *Commun. Math. Phys.*, **80**, 555–561 (1981).
- [70] J. Fröhlich, Verification of axioms for Euclidean and relativistic fields and Haag’s theorem in a class of $P(\phi)_2$ models, *Ann. Inst. Henri Poincaré*, **21**, 271–317 (1974).
- [71] J. Fröhlich, New super-selection sectors (“soliton states”) in two dimensional Bose quantum field models, *Commun. Math. Phys.*, **47**, 269–310 (1976).
- [72] J. Fröhlich, On the triviality of $\lambda\phi_4^4$ theories and the approach to the critical point in $d \geq 4$ dimensions, *Nucl. Phys.*, **B200**, 281–296 (1982).
- [73] J. Fröhlich and E. Seiler, The massive Thirring–Schwinger model (QED_2): Convergence of perturbation theory and particle structure, *Helv. Phys. Acta*, **49**, 889–924 (1976).
- [74] J. Fröhlich and R. Marchetti, Bosonization, topological solitons and fractional charges in two-dimensional quantum field theory, *Commun. Math. Phys.*, **116**, 127–173 (1988).
- [75] K. Gawedzki and A. Kupiainen, Gross–Neveu model through convergent perturbation expansions, *Commun. Math. Phys.*, **102**, 1–30 (1985).
- [76] K. Gawedzki and A. Kupiainen, Nontrivial continuum limit of a ϕ_4^4 model with negative coupling constant, *Nucl. Phys.*, **B 257**, 474–504 (1985).
- [77] C. Gérard and C.D. Jäkel, Thermal quantum fields without cut-offs in 1+1 space-time dimensions, *Rev. Math. Phys.*, **17**, 113–173 (2005).
- [78] C. Gérard and C.D. Jäkel, On the relativistic KMS condition for the $P(\phi)_2$ model, in: *Rigorous Quantum Field Theory*, ed. by A.B. de Monvel *et al.* (Birkhäuser, Basel) 2007, pp. 125–140.
- [79] R. Gielerak and P. Lugiewicz, 4D local quantum field theory models from covariant stochastic differential equations I. Generalities, *Rev. Math. Phys.*, **13**, 335–408 (2001).
- [80] V. Glaser, An explicit solution of the Thirring model, *Nuovo Cim.*, **9**, 990–1006 (1958).
- [81] J. Glimm and A. Jaffe, A $\lambda\phi^4$ quantum theory without cutoffs, I, *Phys. Rev.*, **176**, 1945–1951 (1968).
- [82] J. Glimm and A. Jaffe, A $\lambda\phi^4$ quantum theory without cutoffs, II: The field operators and the approximate vacuum, *Ann. Math.*, **91**, 362–401 (1970).
- [83] J. Glimm and A. Jaffe, A $\lambda\phi^4$ quantum theory without cutoffs, III: The physical vacuum, *Acta Math.*, **125**, 203–267 (1970).
- [84] J. Glimm and A. Jaffe, A $\lambda\phi^4$ quantum theory without cutoffs, IV: Perturbations of the Hamiltonian, *J. Math. Phys.*, **13**, 1568–1584 (1972).
- [85] J. Glimm and A. Jaffe, Self-Adjointness of the Yukawa₂ Hamiltonian, *Ann. Phys.*, **60**, 321–383 (1970).

- [86] J. Glimm and A. Jaffe, The energy momentum spectrum and vacuum expectation values in quantum field theory, *J. Math. Phys.*, **11**, 3335–3338 (1970).
- [87] J. Glimm and A. Jaffe, The energy momentum spectrum and vacuum expectation values in quantum field theory, II, *Commun. Math. Phys.*, **22**, 1–22 (1971).
- [88] J. Glimm and A. Jaffe, Positivity and Self Adjointness of the $P(\phi)_2$ Hamiltonian, *Commun. Math. Phys.*, **22**, 253–258 (1971).
- [89] J. Glimm and A. Jaffe, The Yukawa₂ quantum field theory without cutoffs, *J. Funct. Anal.*, **7**, 323–357 (1971).
- [90] J. Glimm and A. Jaffe, Boson quantum field theory models, in: *Mathematics of Contemporary Physics*, ed. by R.F. Streater (Academic Press, London) 1972, pp. 77–143.
- [91] J. Glimm and A. Jaffe, Positivity of the ϕ_3^4 Hamiltonian, *Fort. Phys.*, **21**, 327–376 (1973).
- [92] J. Glimm, A. Jaffe and T. Spencer, The Wightman axioms and particle structure in the $P(\phi)_2$ quantum field model, *Ann. Math.*, **100**, 585–632 (1974).
- [93] J. Glimm, A. Jaffe and T. Spencer, Phase transitions for ϕ_2^4 quantum fields, *Commun. Math. Phys.*, **45**, 203–216 (1975).
- [94] J. Glimm, A. Jaffe and T. Spencer, A convergent expansion about mean field theory I. The expansion, *Ann. Phys.*, **101**, 610–630 (1976); A convergent expansion about mean field theory II. Convergence of the expansion, *ibid.*, 631–669 (1976).
- [95] J. Glimm and A. Jaffe, *Quantum Physics. A Functional Integral Point of View* (Springer Verlag, Berlin, Heidelberg and New York) 1987.
- [96] D. Gross and A. Neveu, Dynamical symmetry breaking in asymptotically free field theories, *Phys. Rev.*, **D10**, 3235–3253 (1974).
- [97] L. Gross, Convergence of $U(1)_3$ lattice gauge theory to the continuum limit, *Commun. Math. Phys.*, **92**, 137–162 (1983).
- [98] L. Gross, C. King and A. Sengupta, Two-dimensional Yang–Mills theory via stochastic differential equations, *Ann. Phys.*, **194**, 65–112 (1989).
- [99] H. Grosse and G. Lechner, Wedge-local quantum fields and noncommutative Minkowski space, *JHEP*, **0711**, 012 (2007).
- [100] H. Grosse and G. Lechner, Noncommutative deformations of Wightman quantum field theories, *JHEP*, **0809**, 131 (2008).
- [101] H. Grundling and G. Rudolph, QCD on an infinite lattice, preprint arXiv:1108.2129.
- [102] F. Guerra, L. Rosen and B. Simon, The $P(\phi)_2$ Euclidean quantum field theory as classical statistical mechanics, *Ann. Math.*, **101**, 111–259 (1975).
- [103] R. Gurau, J. Magnen and V. Rivasseau, Tree quantum field theory, *Ann. Henri Poincaré*, **10**, 867–891 (2009).
- [104] R. Haag, *Local Quantum Physics* (Springer Verlag, Berlin, Heidelberg and New York) 1992.
- [105] W. Heisenberg and W. Pauli, Zur Quantendynamik der Wellenfelder, *Z. f. Phys.*, **56**, 1–61 (1929).

- [106] W. Heisenberg and W. Pauli, Zur Quantendynamik der Wellenfelder, II, *Z. f. Phys.*, **59**, 168–190 (1930).
- [107] R. Hoegh-Krohn, A General class of quantum fields without cut-offs in two space-time dimensions, *Commun. Math. Phys.*, **21**, 244–255 (1971).
- [108] R. Hoegh-Krohn, Relativistic quantum statistical mechanics in two-dimensional space-time, *Commun. Math. Phys.*, **38**, 195–224 (1974).
- [109] S.S. Horuzhy, *Introduction to Algebraic Quantum Field Theory* (Kluwer Academic Publ., Dordrecht) 1986.
- [110] D. Iagolnitzer and J. Magnen, Bethe–Salpeter kernel and short distance expansion in the massive Gross–Neveu model, *Commun. Math. Phys.*, **119**, 567–584 (1988).
- [111] D. Iagolnitzer, *Scattering in Quantum Field Theories* (Princeton University Press, Princeton, NJ) 1993.
- [112] V.A. Il'in and D.A. Slavnov, Algebras of observables in the S-matrix approach, *Theor. Math. Phys.*, **36**, 578–585 (1978).
- [113] J.Z. Imbrie, Phase diagrams and cluster expansions for low temperature $P(\phi)_2$ models, I, The Phase diagram *Commun. Math. Phys.*, **82**, 261–304 (1981).
- [114] L. Jakobczyk and F. Strocchi, Euclidean formulation of quantum field theory without positivity, *Commun. Math. Phys.*, **119**, 529–541 (1988).
- [115] C.D. Jäkel and F. Robl, The relativistic KMS condition for the thermal n -point functions of the $P(\phi)_2$ model, preprint.
- [116] A. Jaffe, *Dynamics of a Cutoff $\lambda\phi^4$ Field Theory*, Ph.D. Dissertation, Princeton University 1965.
- [117] A. Jaffe, Divergence of perturbation theory for bosons, *Commun. Math. Phys.*, **1**, 127–149 (1965).
- [118] K. Johnson, Solution of the equations for the Green's functions of a two dimensional relativistic field theory, *Nuovo Cim.*, **20**, 773–790 (1961).
- [119] G. Jona-Lasinio and P.K. Mitter, On the stochastic quantization of field theory, *Commun. Math. Phys.*, **101**, 409–436 (1985).
- [120] R. Jost, *General Theory of Quantized Fields* (American Mathematical Society, Providence, RI) 1965.
- [121] M. Karowski and P. Weisz, Exact form factors in $(1 + 1)$ -dimensional field theoretic models with soliton behaviour, *Nucl. Phys. B*, **139**, 455–476 (1978).
- [122] C. King, The $U(1)$ Higgs model, I. The Continuum limit, *Commun. Math. Phys.*, **102**, 649–677 (1986).
- [123] C. King, The $U(1)$ Higgs model, II. The Infinite volume limit, *Commun. Math. Phys.*, **103**, 323–349 (1986).
- [124] B. Klaiber, The Thirring model, in: *Quantum Theory and Statistical Physics*, Vol. X, ed. by A.O. Barut and W.E. Brittin (Gordon and Breach, New York) 1968, pp. 141–176.

- [125] J. Klauder, *A Modern Approach to Functional Integration* (Birkhäuser Verlag, New York) 2011.
- [126] J. Klauder, Scalar field quantization without divergences in all spacetime dimensions, *J. Phys. A: Math. Theor.*, **44**, 273001 (2011).
- [127] A. Klein and L. Landau, Stochastic processes associated with KMS states, *J. Funct. Anal.*, **42**, 368–428 (1981).
- [128] S. Klimek and W. Kondracki, A construction of two-dimensional quantum chromodynamics, *Commun. Math. Phys.*, **113**, 389–402 (1987).
- [129] H. Koch, Particles exist in the low temperature ϕ_2^4 model, *Helv. Phys. Acta*, **53**, 429–452 (1980).
- [130] C. Kopper, J. Magnen and V. Rivasseau, Mass generation in the large N Gross–Neveu model, *Commun. Math. Phys.*, **169**, 121–180 (1995).
- [131] C. Kopper, On the local Borel transform of perturbation theory, *Commun. Math. Phys.*, **295**, 669–699 (2010).
- [132] O.E. Lanford, *Construction of Quantum Fields Interacting by a Cutoff Yukawa Coupling*, Ph.D. Dissertation, Princeton University 1965.
- [133] G. Lechner, Polarization-free quantum fields and interaction, *Lett. Math. Phys.*, **64**, 137–154 (2003).
- [134] G. Lechner, On the existence of local observables in theories with a factorizing S–matrix, *J. Phys. A*, **38**, 3045–3056 (2005).
- [135] G. Lechner, Construction of quantum field theories with factorizing S–matrices, *Commun. Math. Phys.*, **277**, 821–860 (2008).
- [136] G. Lechner, Deformations of quantum field theories and integrable models, preprint arXiv:1104.1948.
- [137] A. Lesniewski, Effective action for the Yukawa₂ quantum field theory, *Commun. Math. Phys.*, **108**, 437–467 (1987).
- [138] J. Magnen and R. Sénéor, The infinite volume limit of the ϕ_3^4 model, *Ann. Inst. Henri Poincaré*, **24**, 95–159 (1976).
- [139] J. Magnen and R. Sénéor, The Wightman axioms for the weakly coupled Yukawa model in two dimensions, *Commun. Math. Phys.*, **51**, 297–313 (1976).
- [140] J. Magnen and R. Sénéor, Phase space cell expansion and Borel summability for the Euclidean ϕ_3^4 , *Commun. Math. Phys.*, **56**, 237–276 (1977).
- [141] J. Magnen and R. Sénéor, Yukawa quantum field theory in three dimensions, in: *Third International Conference on Collective Phenomena (Moscow, 1978)*, *Ann. New York Acad. Sci.*, **337**, 13–43 (1980).
- [142] J. Magnen, V. Rivasseau and R. Sénéor, Construction of YM₄ with an infrared cutoff, *Commun. Math. Phys.*, **155**, 325–383 (1993).
- [143] J. Magnen and V. Rivasseau, Constructive ϕ^4 field theory without tears, *Ann. Henri Poincaré*, **9**, 403–424 (2008).

- [144] B.M. McCoy, C.A. Tracy, and T.T. Wu, Two-dimensional Ising model as an exactly solvable relativistic quantum field theory: Explicit formulas for n-point functions, *Phys. Rev. Lett.*, **38**, 793–796 (1977).
- [145] J. Mund, B. Schroer and J. Yngvason, String-localized quantum fields and modular localization, *Commun. Math. Phys.*, **268**, 621–672 (2006).
- [146] E. Nelson, Probability theory and Euclidean field theory, in: *Constructive Quantum Field Theory*, ed. by G. Velo and A.S. Wightman (Springer-Verlag, New York) 1973, pp. 94–124.
- [147] R. Neves da Silva, Three-particle bound states in even $\lambda P(\phi)_2$, *Helv. Phys. Acta*, **54**, 131–190 (1981/82).
- [148] F. Nicolò, J. Renn and A. Steinmann, On the massive Sine–Gordon equation in all regions of collapse, *Commun. Math. Phys.*, **105**, 291–326 (1986).
- [149] F. Nicolò and P. Perfetti, The sine-Gordon field theory model at $\alpha^2 = 8\pi$, the nonsuper-renormalizable theory, *Commun. Math. Phys.*, **123**, 425–452 (1989).
- [150] I. Ojima, Lorentz invariance vs. temperature in QFT, *Lett. Math. Phys.*, **11**, 73–80 (1986).
- [151] K. Osterwalder and R. Schrader, Euclidean Fermi fields and a Feynman-Kac formula for boson–fermion interactions, *Helv. Phys. Acta*, **46**, 227–302 (1973).
- [152] K. Osterwalder and R. Schrader, Axioms for Euclidean Green’s functions, II, *Commun. Math. Phys.*, **42**, 281–305 (1975).
- [153] K. Osterwalder and R. Sénéor, The scattering matrix is non-trivial for weakly coupled $P(\phi)_2$ models, *Helv. Phys. Acta*, **49**, 525–535 (1976).
- [154] K. Osterwalder and E. Seiler, Gauge field theories on a lattice, *Ann. Phys.*, **110**, 440–471 (1978).
- [155] Y.M. Park, Convergence of lattice approximations and infinite volume limit in the $(\lambda\phi^4 - \sigma\phi^2 - \mu\phi)_3$ field theory, *J. Math. Phys.*, **18**, 354–366 (1977).
- [156] Y.M. Park, Massless quantum sine–Gordon equation in two space-time dimensions: Correlation inequalities and infinite volume limit, *J. Math. Phys.*, **18**, 2423–2426 (1977).
- [157] P. Renouard, Analyticité et sommabilité ”de Borel” des fonctions de Schwinger du modèle de Yukawa en dimension $d = 2$. II. La ”limite adiabatique”, *Ann. Inst. H. Poincar*, **31**, 235–318 (1979).
- [158] V. Rivasseau, *From Perturbative to Constructive Renormalization* (Princeton University Press, Princeton, NJ) 1991.
- [159] S.N.M. Ruijsenaars, The Wightman axioms for the fermionic Federbush model, *Commun. Math. Phys.*, **87**, 181–228 (1982).
- [160] S.N.M. Ruijsenaars, Scattering theory for the Federbush, massless Thirring and continuum Ising models, *J. Funct. Anal.*, **48**, 135–171 (1982).
- [161] S.N.M. Ruijsenaars, On the two-point functions of some integrable relativistic quantum field theories, *J. Math. Phys.*, **24**, 922–931 (1983).
- [162] D. Schlingemann, Construction of kink sectors for two-dimensional quantum field theory models: An algebraic approach, *Rev. Math. Phys.*, **10**, 851–891 (1998).

- [163] A. U. Schmidt, Euclidean reconstruction in quantum field theory: Between tempered distributions and Fourier hyperfunctions, preprint arXiv:math-ph/9811002.
- [164] R. Schrader, A Remark on Yukawa plus Boson selfinteraction in two spacetime dimensions, *Commun. Math. Phys.*, **21**, 164–170 (1971).
- [165] R. Schrader, Yukawa quantum field theory in two spacetime dimensions without cutoffs, *Ann. Phys.*, **70**, 412–457 (1972).
- [166] R. Schrader, Local operator products and field equations in $P(\phi)_2$ theories, *Fortschr. d. Phys.*, **22**, 611–631 (1974).
- [167] B. Schroer, Modular localization and the bootstrap–formfactor program, *Nucl. Phys. B*, **499**, 547–568 (1997).
- [168] B. Schroer, Modular wedge localization and the $d = 1 + 1$ formfactor program, *Ann. Phys.*, **275**, 190–223 (1999).
- [169] S.S. Schweber, *QED and the Men Who Made It* (Princeton University Press, Princeton, NJ) 1994.
- [170] E. Seiler, Schwinger functions for the Yukawa model in two dimensions with space-time cutoff, *Commun. Math. Phys.*, **42**, 163–182 (1975).
- [171] E. Seiler and B. Simon, Nelson’s symmetry and all that in the Yukawa₂ and ϕ_3^4 field theories, *Ann. Math.*, **97**, 470–518 (1976).
- [172] E. Seiler, *Gauge Theories as a Problem of Constructive Quantum Field Theory and Statistical Mechanics*, Lecture Notes in Physics, Vol. 159 (Springer-Verlag, Berlin) 1982.
- [173] B. Simon and R. Griffiths, The ϕ_2^4 field theory as a classical Ising model, *Commun. Math. Phys.*, **33**, 145–164 (1973).
- [174] B. Simon, *The $P(\phi)_2$ (Quantum) Field Theory* (Princeton University Press, Princeton) 1974.
- [175] G. Slade, The loop expansion for the effective potential in the $P(\phi)_2$ quantum field theory, *Commun. Math. Phys.*, **102**, 425–462 (1985).
- [176] F. A. Smirnov, *Form Factors in Completely Integrable Models of Quantum Field Theory* (World Scientific Publishing, River Edge, NJ) 1992.
- [177] T. Spencer and F. Zirilli, Scattering states and bound states in $\lambda P(\phi)_2$, *Commun. Math. Phys.*, **49**, 1–16 (1976).
- [178] F. Strocchi and A.S. Wightman, Proof of the charge superselection rule in local relativistic quantum field theory, *J. Math. Phys.*, **15**, 2198–2224 (1974); Erratum, *ibid.*, **17**, 1930–1931 (1976).
- [179] R.F. Streater and A.S. Wightman, *PCT, Spin and Statistics, and All That* (Reading, Mass., Benjamin/Cummings Publ. Co.) 1964.
- [180] F. Strocchi, *Selected Topics on the General Properties of Quantum Field Theory* (World Scientific, Singapore) 1993.
- [181] S.J. Summers, On the phase diagram of a $P(\phi)_2$ quantum field model, *Ann. Inst. Henri Poincaré*, **34**, 173–229 (1981).

- [182] S.J. Summers, Normal product states for fermions and twisted duality for CCR- and CAR-type algebras with application to the Yukawa₂ quantum field model, *Commun. Math. Phys.*, **86**, 111–141 (1982).
- [183] S.J. Summers, From algebras of local observables to quantum fields: Generalized H -bounds, *Helv. Phys. Acta*, **60**, 1004–1023 (1987).
- [184] S.J. Summers, On the Stone–von Neumann uniqueness theorem and its ramifications, in: *John von Neumann and the Foundations of Quantum Physics*, edited by M. Rédei and M. Stoezlner (Dordrecht, Kluwer Academic Publishers), 2001, pp. 135–152.
- [185] S.J. Summers, Tomita-Takesaki modular theory, in: Volume 5 of the *Encyclopedia of Mathematical Physics*, edited by J.-P. Francoise, G. Naber and T.S. Tsun, 2006, pp. 251–257.
- [186] S.J. Summers, Yet more ado about nothing: The remarkable relativistic vacuum state, in: *Deep Beauty*, ed. by H. Halvorson (Cambridge University Press, Cambridge), 2011, pp. 317–341.
- [187] K. Symanzik, Euclidean quantum field theory, I. Equations for a scalar model, *J. Math. Phys.*, **7**, 510–525 (1966).
- [188] K. Symanzik, Euclidean quantum field theory, in: *Local Quantum Theory*, edited by R. Jost (Academic Press, New York) 1969, pp. 152–226.
- [189] M. Takesaki, *Theory of Operator Algebras*, Volume II (Springer Verlag, Berlin, Heidelberg and New York) 2003.
- [190] T. Thiemann, An axiomatic approach to quantum gauge field theory, in: *Symplectic Singularities and Geometry of Gauge Fields* (Warsaw, 1995), *Banach Center Publ.*, **39** (Polish Academy of Science, Warsaw) 1997, pp. 389–403.
- [191] W. Thirring, A soluble relativistic field theory, *Ann. Phys.*, **3**, 91–112 (1958).
- [192] S. Waldmann, *Poisson-Geometrie und Deformationsquantisierung* (Springer-Verlag, Berlin, Heidelberg, New York) 2007.
- [193] D.H. Weingarten and J.L. Challifour, Continuum limit of QED₂ on a lattice, *Ann. Phys.*, **123**, 61–101 (1979).
- [194] D. Weingarten, Continuum limit of QED₂ on a lattice, II, *Ann. Phys.*, **126**, 154–175 (1980).
- [195] E. Wigner, On unitary representations of the inhomogeneous Lorentz group, *Ann. Math.*, **40**, 149–204 (1939).
- [196] A.S. Wightman and L. Gårding, Fields as operator-valued distributions in relativistic quantum field theory, *Ark. f. Fys.*, **28**, 129–184 (1964).
- [197] K.G. Wilson, Confinement of quarks, *Phys. Rev.*, **D 10**, 2445–2459 (1974).
- [198] L. Wu, Uniqueness of Nelson’s diffusions II: Infinite dimensional setting and applications, *Potential Analysis*, **13**, 269–301 (2000).
- [199] A.B. Zamolodchikov and A.B. Zamolodchikov, Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field models, *Ann. Phys.*, **120**, 253–291 (1979).